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Bounded trajectories on homogeneous spaces for subsets without semigroup structure

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The homogeneous setup

$$X = G/\Gamma, \quad g_t = \exp(t\mathbf{v}), \quad F = \{g_t : t \geq 0\}.$$

- G is connected semisimple, without compact factors.
- Γ is an irreducible lattice.
- $\mathrm{ad} \mathbf{v}$ is diagonalizable over $\mathbb{R}$, and $\mathbf{v} \neq 0$.

Normalize Haar measure: $\mu_X(X) = 1$. The flow is mixing, hence ergodic.

The expanding horospherical subgroup is

$$U = U_+(g_1) := \{u \in G : g_{-t}ug_t \longrightarrow e \text{ as } t \rightarrow \infty\}.$$

Bounded trajectories have measure zero

Write

$$\mathcal{B}(F) = \{x \in X : Fx \text{ is relatively compact}\}.$$

Assume X is **noncompact**.

Birkhoff's theorem implies that almost every trajectory is dense, so

$$\mu_X(\mathcal{B}(F)) = 0.$$

The same zero-measure conclusion holds on every expanding leaf:

$$\mu_U\{u \in U : ux \in \mathcal{B}(F)\} = 0 \quad \text{for every } x \in X.$$

How large can the exceptional set still be?

Full Hausdorff dimension on every expanding leaf

Theorem (Kleinbock–Margulis, 1996)

For every $x \in X$ and every nonempty open $V \subset U$,

$$\dim_H\{u \in V : ux \in \mathcal{B}(F)\} = \dim U.$$

The set is **thick in U** : it has full Hausdorff dimension locally.

Also, for every nonempty open $W \subset X$,

$$\dim_H(W \cap \mathcal{B}(F)) = \dim X.$$

The compact set containing a trajectory may depend on its starting point.

Approximating two numbers with one denominator

Fix $\Theta = (\theta_1, \theta_2) \in \mathbb{R}^2$. We approximate it by

$$\left(\frac{p_1}{q}, \frac{p_2}{q} \right), \quad p_1, p_2 \in \mathbb{Z}, \quad q \in \mathbb{N}.$$

Write $\|y\| = \min_{p \in \mathbb{Z}} |y - p|$ for distance to the nearest integer.

Theorem (Asymptotic version of Dirichlet's theorem)

For every $\Theta \in \mathbb{R}^2$, infinitely many $q \in \mathbb{N}$ satisfy

$$\max\{\|q\theta_1\|, \|q\theta_2\|\} \leq \frac{1}{\sqrt{q}}.$$

Equivalently, both rational-approximation errors are at most $q^{-3/2}$.

Badly approximable pairs

An arbitrarily small Dirichlet constant is not possible for every pair

A pair $\Theta = (\theta_1, \theta_2)$ is **badly approximable** if there is $c > 0$ such that, for every $q \in \mathbb{N}$,

$$\max\{\|q\theta_1\|, \|q\theta_2\|\} \geq \frac{c}{\sqrt{q}}.$$

Write $\mathbf{BA} \subset \mathbb{R}^2$ for this set. Equivalently,

$$\Theta \in \mathbf{BA} \iff \inf_{q \in \mathbb{N}} \sqrt{q} \max\{\|q\theta_1\|, \|q\theta_2\|\} > 0.$$

Pairs as lattices in three dimensions

$$X_3 = \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z}) = \{\text{unimodular lattices in } \mathbb{R}^3\}.$$

$$u_{\Theta} = \begin{pmatrix} 1 & 0 & \theta_1 \\ 0 & 1 & \theta_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_t = \mathrm{diag}(e^{t/2}, e^{t/2}, e^{-t}).$$

One has

$$g_t u_{\Theta} \begin{pmatrix} -p_1 \\ -p_2 \\ q \end{pmatrix} = \begin{pmatrix} e^{t/2}(q\theta_1 - p_1) \\ e^{t/2}(q\theta_2 - p_2) \\ e^{-t}q \end{pmatrix}.$$

Here $U = \{u_{\Theta} : \Theta \in \mathbb{R}^2\} = U_+(g_1)$.

Dani correspondence

Mahler's compactness criterion

A family of unimodular lattices is relatively compact exactly when all nonzero lattice vectors have a common positive lower bound in norm.

Theorem (Dani)

$$\Theta \in \mathbf{BA} \iff \{g_t u_\Theta \mathbb{Z}^3 : t \geq 0\} \text{ is bounded in } X_3.$$

Badly approximable pairs are thick

The map $\Theta \mapsto u_\Theta$ identifies $\mathbb{R}^2$ with U .

Dani correspondence and the dynamical results give

$$\text{Leb}(\mathbf{BA}) = 0, \quad \dim_H(\mathbf{BA} \cap V) = 2$$

for every nonempty open $V \subset \mathbb{R}^2$.

Thus badly approximable pairs form a **thick set of measure zero**.

Remark

The zero-measure result already follows from the classical Khintchine–Groshev theorem. Schmidt (1969) proved the winning property, which implies full Hausdorff dimension.

Weight functions in the two coordinates

For a denominator $q \in \mathbb{N}$, allow different error bounds:

$$\|q\theta_1\| \leq \alpha_1(1/q), \quad \|q\theta_2\| \leq \alpha_2(1/q).$$

The weight functions $\alpha_1, \alpha_2 : (0, 1] \rightarrow (0, 1]$ are increasing bijections satisfying

$$\alpha_1(s)\alpha_2(s) = s.$$

Cases studied before

The classical choice is $\alpha_1(s) = \alpha_2(s) = \sqrt{s}$.

Fixed weights give $\alpha_1(s) = s^r$, $\alpha_2(s) = s^{1-r}$, where $0 < r < 1$.

Dirichlet's theorem with two weight functions

Theorem

For every $\Theta \in \mathbb{R}^2$, infinitely many $q \in \mathbb{N}$ satisfy

$$\begin{cases} \|q\theta_1\| \leq \alpha_1(1/q), \\ \|q\theta_2\| \leq \alpha_2(1/q). \end{cases}$$

The normalization is

$$\alpha_1(1/q)\alpha_2(1/q) = \frac{1}{q}.$$

The available accuracy can be distributed unevenly between the coordinates.

Bad approximation with weight functions

Write $\vec{\alpha} = (\alpha_1, \alpha_2)$. A pair $\Theta \in \mathbb{R}^2$ belongs to $\mathbf{BA}_{\vec{\alpha}}$ if there is $c \in (0, 1)$ such that, for every $q \in \mathbb{N}$,

$$\max \left\{ \frac{\|q\theta_1\|}{\alpha_1(c/q)}, \frac{\|q\theta_2\|}{\alpha_2(c/q)} \right\} > 1.$$

At every denominator, at least one error exceeds the Dirichlet bound improved by a constant.

The improvement constant sits inside the functions.

For $\alpha_1(s) = \alpha_2(s) = \sqrt{s}$, this recovers $\mathbf{BA}$.

Controlled change of scale

Both weight functions are **quasimultiplicative (QM)** if, for some $R_0 > 1$ and $0 < \kappa_1 < \kappa_2$,

$$R^{\kappa_1} \alpha_i(s) \leq \alpha_i(Rs) \leq R^{\kappa_2} \alpha_i(s), \quad i = 1, 2,$$

for $R \geq R_0$.

Examples

1. **Power functions:** $\alpha_i(s) = s^{\tau_i}$, $\tau_i > 0$, $\tau_1 + \tau_2 = 1$.
2. **Differentiable weight functions** with

$$0 < \kappa_1 \leq \frac{s\alpha'_i(s)}{\alpha_i(s)} \leq \kappa_2, \quad i = 1, 2.$$

A diagonal curve instead of a flow

Define $F^{\vec{\alpha}} = \{g(t) : t \geq 0\}$, where

$$g(t) = \text{diag}\left(\frac{1}{\alpha_1(e^{-t})}, \frac{1}{\alpha_2(e^{-t})}, e^{-t}\right).$$

The normalization gives $\det g(t) = 1$.

Theorem (Dani correspondence under QM)

$$\Theta \in \mathbf{BA}_{\vec{\alpha}} \iff F^{\vec{\alpha}} u_{\Theta} \mathbb{Z}^3 \text{ is bounded in } X_3.$$

In general, $g(t+s) \neq g(t)g(s)$.

The approximation problem leads to a set without semigroup structure.

The curve in a two-dimensional diagonal plane

Let A be the positive diagonal subgroup of $\mathrm{SL}_3(\mathbb{R})$. Its Lie algebra is

$$\mathfrak{a} = \{\mathrm{diag}(a_1, a_2, -a_1 - a_2) : a_1, a_2 \in \mathbb{R}\}.$$

Writing $a_i(t) = -\log \alpha_i(e^{-t})$, we have

$$\mathbf{v}(t) = \log g(t) = \mathrm{diag}(a_1(t), a_2(t), -t), \quad a_1(t) + a_2(t) = t.$$

We can therefore draw $\mathbf{v}(t)$ using just the coordinates $(a_1(t), a_2(t))$.

The cones for simultaneous approximation

For $\mathbf{v} = \text{diag}(a_1, a_2, -a_1 - a_2)$, conjugation acts on the pair by

$$\exp(\mathbf{v})u_{(\theta_1, \theta_2)}\exp(-\mathbf{v}) = u_{(e^{2a_1+a_2}\theta_1, e^{a_1+2a_2}\theta_2)}.$$

Geometric expansion: both coordinates of U expand

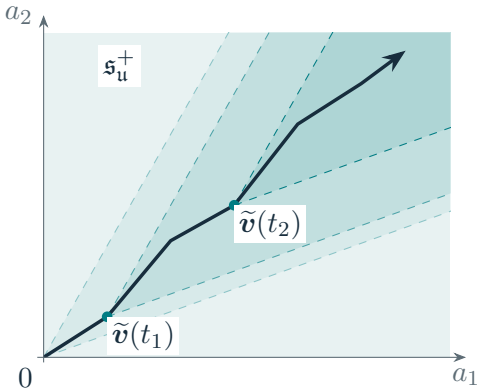
$$\mathfrak{e}_u^+ = \{2a_1 + a_2 > 0, a_1 + 2a_2 > 0\}.$$

The super-expanding cone in this example

$$\mathfrak{s}_u^+ = \{a_1 > 0, a_2 > 0\} \quad (\text{positive quadrant}).$$

Every tail stays in a translated smaller cone

QM keeps long increments away from the walls: they lie in one fixed cone $\mathfrak{s}$ strictly inside $\mathfrak{s}_u^+$.



After sampling and linear interpolation, this holds for every forward increment:

$$\tilde{v}([t, \infty)) \subset \tilde{v}(t) + \bar{\mathfrak{s}}.$$

Here $\bar{\mathfrak{s}} \subset \mathfrak{s}_u^+ \cup \{0\}$.

The original curve stays a bounded distance from $\tilde{v}$.

The super-expanding cone in general

Return to $X = G/\Gamma$ and a nontrivial horospherical subgroup U . Choose a compatible diagonal subgroup A and put $\mathfrak{a} = \text{Lie}(A)$, $\mathfrak{u} = \text{Lie}(U)$.

Let $\Phi_{\mathfrak{u}}$ be the roots occurring in $\text{Lie}(U)$, and let $\mathbf{v}_{\lambda}$ be the vector dual to λ for the Killing inner product.

$$\mathfrak{a}_{\mathfrak{u}}^+ = \left\{ \sum_{\lambda \in \Phi_{\mathfrak{u}}} t_{\lambda} \mathbf{v}_{\lambda} : t_{\lambda} > 0 \right\},$$

$$\mathfrak{e}_{\mathfrak{u}}^+ = \{ \mathbf{v} \in \mathfrak{a} : \lambda(\mathbf{v}) > 0 \text{ for all } \lambda \in \Phi_{\mathfrak{u}} \}.$$

$$\mathfrak{s}_{\mathfrak{u}}^+ = \mathfrak{a}_{\mathfrak{u}}^+ \cap \mathfrak{e}_{\mathfrak{u}}^+.$$

For the pair-approximation subgroup in SL_3 , this is $a_1, a_2 > 0$.

Quasi-rays

Let $\mathfrak{s} \subset \mathfrak{a}$ be a convex cone with

$$\bar{\mathfrak{s}} \subset \mathfrak{s}_u^+ \cup \{0\}.$$

Definition

A sequence $(\mathbf{v}_k)_{k \geq 1}$ in $\mathfrak{s}$, with $\mathbf{v}_0 = 0$, is a **quasi-ray** if

$$\mathbf{v}_k - \mathbf{v}_{k-1} \in \bar{\mathfrak{s}}, \quad 0 < c_0 \leq |\mathbf{v}_k - \mathbf{v}_{k-1}| \leq L < \infty.$$

A set $F \subset G$ is a quasi-ray relative to $\mathfrak{s}$ if, for some such sequence,

$$F \subset C\{\exp(\mathbf{v}_k) : k \geq 1\} \quad \text{for some compact } C \subset G.$$

The diagonal curves coming from QM weight functions have this form.

The main theorem on expanding leaves

Theorem (Kleinbock–Neckrasov)

Let $X = G/\Gamma$ and U, A be as above. Suppose

$$\bar{\mathfrak{s}} \subset \mathfrak{s}_u^+ \cup \{0\}, \quad F \text{ is a quasi-ray relative to } \mathfrak{s}.$$

Then for every $x \in X$ and every nonempty open $V \subset U$,

$$\dim_H \{u \in V : Fux \text{ is bounded}\} = \dim U.$$

The main theorem on the whole space

Under the same assumptions, for every nonempty open $W \subset X$,

$$\dim_H(W \cap \mathcal{B}(F)) = \dim X.$$

Thus the set of bounded trajectories is **thick in X** .

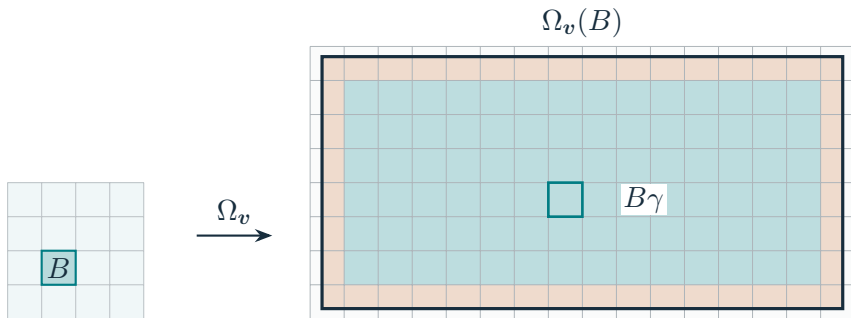
Application to simultaneous approximation of pairs

If α_1, α_2 are QM and $\alpha_1(s)\alpha_2(s) = s$, then

$$\dim_H(\mathbf{BA}_{\vec{\alpha}} \cap V) = 2 \quad \text{for every nonempty open } V \subset \mathbb{R}^2.$$

Role of geometric expansion $\mathfrak{e}_u^+$

$$\Omega_v(B) = \exp(\mathbf{v})B \exp(-\mathbf{v}).$$

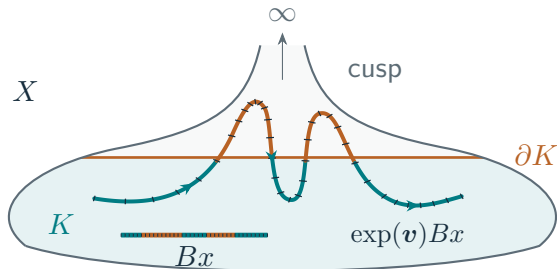


Teal: whole tiles. Orange: boundary loss.

For $\mathbf{v} \in \mathfrak{e}_u^+$, relative boundary loss tends uniformly to 0 as $\text{dist}(\mathbf{v}, \partial \mathfrak{e}_u^+) \rightarrow \infty$.

Role of algebraic expansion $\mathfrak{a}_u^+$: return of mass

Fix a compact $K \subset X$ with $\mu_X(\partial K) = 0$ and $x \in X$.



Keep: segments whose images lie wholly in K .

Discard: segments whose images leave K or cross its boundary.

$$\frac{\mu_U\{u \in B : \exp(v)ux \in K\}}{\mu_U(B)} \longrightarrow \mu_X(K),$$

uniformly over $v \in \mathfrak{s}_u^+$ as $\text{dist}(v, \partial \mathfrak{s}_u^+) \rightarrow \infty$.

Thank you for your attention!