

Winning properties of counterexamples to the uniform Littlewood conjecture

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Diophantine approximation

Let $F(q) \geq 0$ measure the error for an integer q .

Let $\psi : (0, \infty) \rightarrow (0, \infty)$ be non-increasing.

We study the solvability of

$$\begin{cases} F(q) \leq \psi(Q), \\ 0 < |q| \leq Q, \end{cases} \quad q \in \mathbb{Z}.$$

Q bounds the denominator, and $\psi(Q)$ is the allowed error.

Uniform and asymptotic approximation

$$\begin{cases} F(q) \leq \psi(Q), \\ 0 < |q| \leq Q, \end{cases} \quad q \in \mathbb{Z}.$$

Uniform approximation

The system has a solution for *every sufficiently large* Q .

Asymptotic approximation

The system has a solution for *arbitrarily large* Q .

For the examples below, the asymptotic condition is equivalent to

$$F(q) \leq \psi(|q|) \quad \text{for infinitely many } q \in \mathbb{Z} \setminus \{0\}.$$

Dirichlet's theorem for one real number

Write $\langle t \rangle := \min_{n \in \mathbb{Z}} |t - n|$ for the distance to the nearest integer.

Fix $x \in \mathbb{R}$. Here

$$F(q) = \langle qx \rangle, \quad \psi(Q) = \frac{1}{Q}.$$

Uniform Dirichlet

For every $Q \geq 1$, there is an integer q such that

$$\begin{cases} \langle qx \rangle \leq Q^{-1}, \\ 1 \leq q \leq Q. \end{cases}$$

Asymptotic Dirichlet

There are infinitely many integers $q \geq 1$ such that

$$\langle qx \rangle \leq q^{-1}.$$

Equivalently, for suitable $p \in \mathbb{Z}$,

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

Dirichlet's theorem for two real numbers

Fix $x, y \in \mathbb{R}$. Use the *same* denominator for both numbers:

$$F(q) = \max\{\langle qx \rangle, \langle qy \rangle\}, \quad \psi(Q) = Q^{-1/2}.$$

Uniform Dirichlet

For every $Q \geq 1$, there is an integer q such that

$$\begin{cases} \max\{\langle qx \rangle, \langle qy \rangle\} \leq Q^{-1/2}, \\ 1 \leq q \leq Q. \end{cases}$$

Asymptotic Dirichlet

There are infinitely many integers $q \geq 1$ such that

$$\max\{\langle qx \rangle, \langle qy \rangle\} \leq q^{-1/2}.$$

Badly approximable objects

Suppose an error bound works for every object:

$$F(q) \leq \psi(|q|) \quad \text{for infinitely many } q.$$

An object is *badly approximable* at this scale if

$$\boxed{\exists c > 0 \quad F(q) \geq c \psi(|q|) \quad \text{for every } q \neq 0.}$$

Equivalently,

$$\inf_{q \in \mathbb{Z} \setminus \{0\}} \frac{F(q)}{\psi(|q|)} > 0.$$

The universal bound cannot be improved by an arbitrarily small constant.

Badly approximable at the simultaneous scale

Dirichlet gives simultaneous errors of order $q^{-1/2}$. The badly approximable pairs have a uniform lower bound at this scale:

$$\begin{aligned}\mathbf{Bad}\left(\frac{1}{2}, \frac{1}{2}\right) &= \left\{ (x, y) \in \mathbb{R}^2 : \inf_{q \in \mathbb{N}} q^{\frac{1}{2}} \max\{\langle qx \rangle, \langle qy \rangle\} > 0 \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 : \inf_{(A, B) \in \mathbb{Z}^2 \setminus \{(0, 0)\}} \max\{|A|, |B|\}^2 \langle Ax + By \rangle > 0 \right\}.\end{aligned}$$

A winning set that will return in the curve proof

This set is HAW in $\mathbb{R}^2$ and on every regular C^2 curve of nowhere-zero curvature.

The games will be introduced shortly.

Plane: Nesharim–Simmons (2014). Curves: An–Beresnevich–Velani (2018), combined with Badziahin–Harrap–Nesharim–Simmons (2025).

Multiplicative asymptotic approximation

Now measure the *product* of the errors:

$$F(q) = \langle qx \rangle \langle qy \rangle, \quad \psi(Q) = \frac{1}{Q}.$$

A consequence of simultaneous Dirichlet

For every $x, y \in \mathbb{R}$, infinitely many integers $q \geq 1$ satisfy

Error-bound form

$$\boxed{\langle qx \rangle \langle qy \rangle \leq \frac{1}{q}}$$

Product form

$$\boxed{q \langle qx \rangle \langle qy \rangle \leq 1}$$

Indeed, $\langle qx \rangle \langle qy \rangle \leq \max\{\langle qx \rangle, \langle qy \rangle\}^2 \leq q^{-1}$ for the denominators given by Dirichlet.

Littlewood's conjecture

A *multiplicatively badly approximable* pair would satisfy

$$\inf_{q \in \mathbb{N}} q \langle qx \rangle \langle qy \rangle > 0.$$

Littlewood, around 1930

For every $(x, y) \in \mathbb{R}^2$,

$$\liminf_{q \rightarrow \infty} q \langle qx \rangle \langle qy \rangle = 0.$$

Littlewood predicts that no multiplicatively badly approximable pair exists.

Unlike the maximum, the product lets one very small error compensate for the other.

The conjecture remains open.

How large could the classical exceptional set be?

Let

$$E' = \left\{ (x, y) \in \mathbb{R}^2 : \liminf_{q \rightarrow \infty} q \langle qx \rangle \langle qy \rangle > 0 \right\}.$$

Einsiedler–Katok–Lindenstrauss (2006)

$$\dim_{\mathrm{H}} E' = 0.$$

Any counterexamples form an exceptionally small set.

The theorem does not establish that E' is empty.

Annals of Mathematics 164 (2006), 513–560.

Uniform Dirichlet revisited

Fix any $x, y \in \mathbb{R}$. Recall

$$F(q) = \max\{\langle qx \rangle, \langle qy \rangle\}, \quad \psi(Q) = Q^{-1/2}.$$

Uniform simultaneous approximation

For every $Q \geq 1$, there is an integer q such that

$$\begin{cases} \max\{\langle qx \rangle, \langle qy \rangle\} \leq Q^{-1/2}, \\ 1 \leq q \leq Q. \end{cases}$$

For this same q ,

$$\langle qx \rangle \langle qy \rangle \leq \max\{\langle qx \rangle, \langle qy \rangle\}^2 \leq \frac{1}{Q}.$$

Multiplicative uniform approximation

Here

$$F(q) = \langle qx \rangle \langle qy \rangle, \quad \psi(Q) = \frac{1}{Q}.$$

For every $x, y \in \mathbb{R}$ and every $Q \geq 1$,

Error-bound form

$$\exists q \in \mathbb{N} : \quad \begin{cases} \langle qx \rangle \langle qy \rangle \leq Q^{-1}, \\ 1 \leq q \leq Q. \end{cases}$$

Product form

$$\boxed{Q \min_{1 \leq q \leq Q} \langle qx \rangle \langle qy \rangle \leq 1}$$

The two statements are equivalent.

The uniform Littlewood question

Bandi–Fregoli–Kleinbock (2025)

Does every $(x, y) \in \mathbb{R}^2$ satisfy

$$Q \min_{1 \leq q \leq Q} \langle qx \rangle \langle qy \rangle \longrightarrow 0 \quad (Q \rightarrow \infty)?$$

Equivalently, for every $\varepsilon > 0$ and all sufficiently large Q ,

$$\exists 1 \leq q \leq Q : \quad \langle qx \rangle \langle qy \rangle < \frac{\varepsilon}{Q}.$$

Bandi–Fregoli–Kleinbock, arXiv:2504.02258.

The uniform conjecture fails

- **Bandi–Fregoli–Kleinbock (2025):**
the uniform statement holds for almost every pair.
- **Schleischitz (2026):**
counterexamples exist and form a residual set.
- **Moshchevitin (2026):**
a simpler construction of counterexamples.

What is the Hausdorff dimension of the counterexamples?

- **Shulga (2026):** Hausdorff dimension $\geq 3/2$.
Independent preprint, submitted on the same day as ours.

Schleischitz, arXiv:2603.12611; Moshchevitin, arXiv:2605.26188; Shulga, arXiv:2608.25059.

Our main theorem

Write

$$E = \left\{ (x, y) \in \mathbb{R}^2 : \limsup_{Q \rightarrow \infty} Q \min_{1 \leq q \leq Q} \langle qx \rangle \langle qy \rangle > 0 \right\}.$$

Neckrasov–Wu–Yang, Theorem 1.2

E is hyperplane absolute winning in $\mathbb{R}^2$.

In particular, E has full Hausdorff dimension 2.

HAW will denote hyperplane absolute winning.

We will define the game after stating the other results.

A prescribed sequence of scales

Let $\mathbf{Q} = (Q_m)_{m \geq 0}$ be strictly increasing, with $Q_0 > 1$ and $Q_m \rightarrow \infty$.

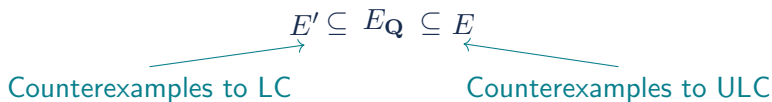
$$E_{\mathbf{Q}} = \left\{ (x, y) : \liminf_{m \rightarrow \infty} Q_m \min_{1 \leq q \leq Q_m} \langle qx \rangle \langle qy \rangle > 0 \right\}.$$

Theorem 1.6

If, for some $\tau > 1$,

$$\inf_{m \geq 0} \frac{Q_{m+1}}{Q_m^\tau} > 0,$$

then $E_{\mathbf{Q}}$ is HAW in $\mathbb{R}^2$.



Sparse scales and classical Littlewood

Theorem 1.6 gives a positive liminf along *any prescribed sufficiently sparse sequence*.

If we used every integer scale, the earlier equivalence would give

$$\liminf_{\substack{Q \rightarrow \infty \\ Q \in \mathbb{N}}} Q \min_{1 \leq q \leq Q} \langle qx \rangle \langle qy \rangle > 0 \\ \iff \inf_{q \in \mathbb{N}} q \langle qx \rangle \langle qy \rangle > 0.$$

Thus $E_{(2,3,4,\dots)} = E'$. Even one such pair would disprove classical Littlewood.

The HAW conclusion cannot hold at all integer scales

Einsiedler–Katok–Lindenstrauss (2006): $\dim_{\mathbb{H}} E' = 0$.

HAW in $\mathbb{R}^2$ would imply Hausdorff dimension 2.

Results on curves

Let $\mathcal{C} \subset \mathbb{R}^2$ be a regular embedded C^2 curve. Write $\kappa(z)$ for its curvature.

Theorem 1.7

If $\{z \in \mathcal{C} : \kappa(z) \neq 0\}$ is HAW on $\mathcal{C}$, then

$$E \cap \mathcal{C} \quad \text{and} \quad E_{\mathbf{Q}} \cap \mathcal{C} \quad \text{are HAW on } \mathcal{C}.$$

Here $\mathbf{Q}$ satisfies the preceding growth assumption.

Concrete sufficient condition:

$$\dim_{\mathbf{H}}\{z \in \mathcal{C} : \kappa(z) = 0\} = 0.$$

In particular: nowhere-zero curvature, or only countably many zeros.

Results on lines

Consider $L_{a,b} = \{(x, ax + b) : x \in \mathbb{R}\}$.

Theorem 1.8

Suppose $a \neq 0$ and, for some $\varepsilon > 0$,

$$\liminf_{q \rightarrow \infty} q^{2-\varepsilon} \max\{\langle qa \rangle, \langle qb \rangle\} > 0.$$

Then $E \cap L_{a,b}$ and $E_{\mathbb{Q}} \cap L_{a,b}$ are HAW on $L_{a,b}$.

An-Beresnevich-Velani (2018) use this same condition for winning of $\mathbf{Bad}(\frac{1}{2}, \frac{1}{2}) \cap L_{a,b}$ on $L_{a,b}$. For example, this includes every badly approximable slope a and every b .

An obstruction

If $a, b \in \mathbb{Q}$, then $L_{a,b} \cap E = \emptyset$.

Schmidt's game

Fix $\alpha, \beta \in (0, 1)$.

$$B_0 \supset A_0 \supset B_1 \supset A_1 \supset \cdots$$

Bob chooses B_n . Alice chooses A_n .

$$\text{rad}(A_n) = \alpha \text{rad}(B_n),$$

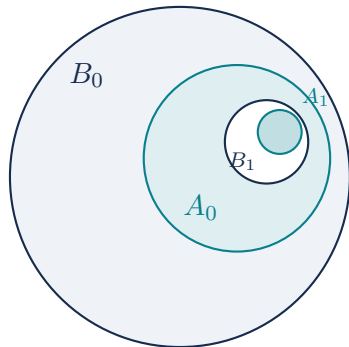
$$\text{rad}(B_{n+1}) = \beta \text{rad}(A_n).$$

Alice wins if the unique limiting point lies in the target set S .

S is α -winning if Alice can win for every β .

Every $\frac{1}{2}$ -winning set in $\mathbb{R}^d$ has full Hausdorff dimension d .

Schmidt, *On badly approximable numbers and certain games*, Trans. AMS 123 (1966).



The hyperplane absolute game in $\mathbb{R}^2$

Fix $0 < \beta < 1/3$.

$L^{(r)}$: the closed r -neighborhood of a line L .

1. Bob chooses $B_n = B(z_n, \rho_n)$.
2. Alice removes $L_n^{(r_n)}$, where L_n is a line and $r_n \leq \beta \rho_n$.
3. Bob chooses

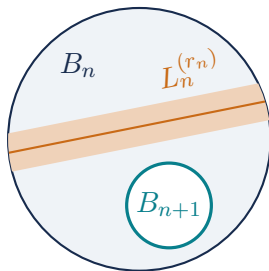
$$B_{n+1} \subseteq B_n \setminus L_n^{(r_n)}, \quad \rho_{n+1} \geq \beta \rho_n.$$

Alice aims for $\bigcap_n B_n \subseteq S$.

S is **HAW** if Alice can win for every $0 < \beta < 1/3$.

Countable-intersection stability: if every S_j is HAW, then $\bigcap_{j \geq 1} S_j$ is HAW.

Broderick–Fishman–Kleinbock–Reich–Weiss (2012). Nonshrinking plays are automatic wins for Alice.



The line potential game in $\mathbb{R}^2$

Fix $\beta \in (0, 1)$ and $\gamma > 0$.

Bob chooses nested balls with $\rho_{n+1} \geq \beta \rho_n$.

Alice marks line neighborhoods $L_{n,i}^{(r_{n,i})}$ subject to

$$\sum_i r_{n,i}^\gamma \leq (\beta \rho_n)^\gamma.$$

Bob need not avoid these neighborhoods.

If $\rho_n \rightarrow 0$, with limiting point z_∞ , Alice wins when

$$z_\infty \in S \quad \text{or} \quad z_\infty \in \bigcup_{n,i} L_{n,i}^{(r_{n,i})}.$$

If the radii do not tend to zero, she wins automatically.

Equivalence of the games

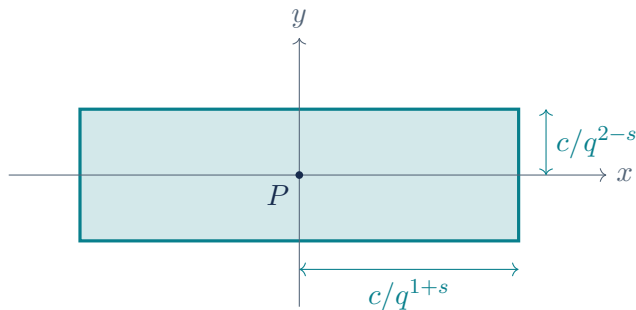
Theorem (Fishman–Simmons–Urbański, 2018)

A set $S \subseteq \mathbb{R}^2$ is HAW if and only if it is (β, γ) -hyperplane potential winning for every $\beta \in (0, 1)$ and every $\gamma > 0$.

A dangerous rectangle around a rational point

Fix $P = (p/q, r/q)$, $c > 0$, and $0 \leq s \leq 1$.

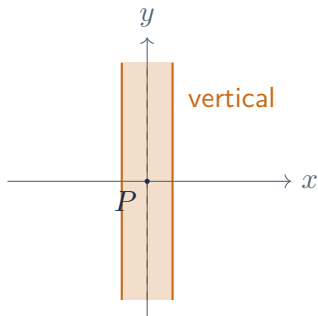
$$\Delta_c(P; s) = \left\{ (x, y) : \left| x - \frac{p}{q} \right| < \frac{c}{q^{1+s}}, \left| y - \frac{r}{q} \right| < \frac{c}{q^{2-s}} \right\}.$$



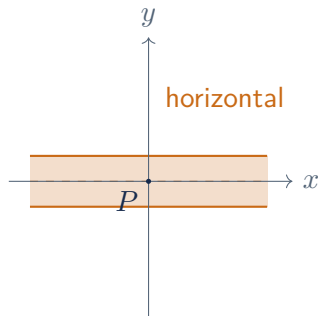
The weight s changes the shape, while the area stays $4c^2/q^3$.

The two coordinate strips

$$\Delta_c^1(P) = \left\{ (x, y) : \left| x - \frac{p}{q} \right| < \frac{c}{q^2} \right\}.$$

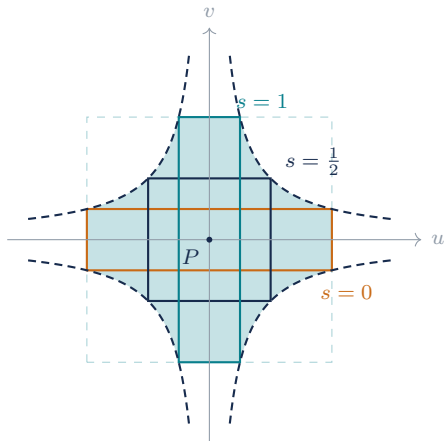


$$\Delta_c^2(P) = \left\{ (x, y) : \left| y - \frac{r}{q} \right| < \frac{c}{q^2} \right\}.$$



One very small coordinate error is allowed, regardless of the other.

The union over all weights



Use the coordinates

$$u = \frac{qx - p}{c}, \quad v = \frac{qy - r}{c}.$$

The rectangle $\Delta_c(P; s)$ satisfies

$$|u| < q^{-s}, \quad |v| < q^{s-1}.$$

The shaded region represents

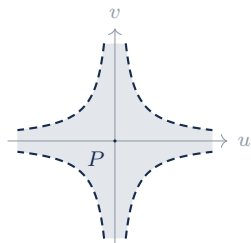
$$\bigcup_{s \in [0,1]} \Delta_c(P; s),$$

which is exactly the region

$$|u| < 1, \quad |v| < 1, \quad |uv| < \frac{1}{q}.$$

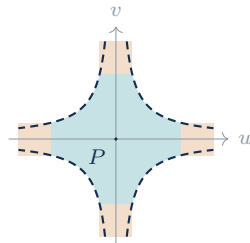
Lemma 3.1: rectangles and strips cover the product region

$$\begin{aligned}
 W_c(P) &:= \{(x, y) : q|qx - p||qy - r| < c^2\} \\
 &= \{(x, y) : |uv| < 1/q\} \subseteq \bigcup_{s \in [0,1]} \Delta_c(P; s) \cup \Delta_c^1(P) \cup \Delta_c^2(P).
 \end{aligned}$$



$$W_c(P) : |uv| < 1/q$$

$\subseteq$



$$\bigcup_{s \in [0,1]} \Delta_c(P; s) \cup \Delta_c^1(P) \cup \Delta_c^2(P)$$

Rectangles cover the central part. The strips cover the remaining arms.

Corollary 3.2: each denominator needs attention only once

Fix $\eta > 0$ and $1 \leq Q_0 < Q_1 < \cdots \rightarrow \infty$. Write

$$c_m(P) = \eta \sqrt{\frac{q}{Q_m}}.$$

Suppose (x, y) satisfies

$$Q_0 \min_{1 \leq q \leq Q_0} \langle qx \rangle \langle qy \rangle \geq \eta^2,$$

and, for every $m \geq 1$ and every rational $P = (p/q, r/q)$ with reduced common denominator $Q_{m-1} < q \leq Q_m$,

$$(x, y) \notin \bigcup_{s \in [0,1]} \Delta_{c_m(P)}(P; s) \cup \Delta_{c_m(P)}^1(P) \cup \Delta_{c_m(P)}^2(P).$$

Then, for every $m \geq 0$,

$$Q_m \min_{1 \leq q \leq Q_m} \langle qx \rangle \langle qy \rangle \geq \eta^2, \quad (x, y) \in E_{\mathbf{Q}} \subseteq E.$$

The simpler clustering statement: Lemma 3.6

Consider thin vertical strips with comparable denominators, all meeting a small ball B .

Distinct rational coordinates satisfy

$$\left| \frac{p_i}{q_i} - \frac{p_j}{q_j} \right| \geq \frac{1}{q_i q_j}.$$

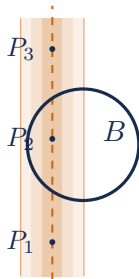
If B and the strips are too small to bridge that separation, then

$$\boxed{\frac{p_i}{q_i} = \frac{p_j}{q_j}}.$$

All strips share one central line.

Many dangerous strips cost only one line neighborhood.

one vertical line



Two scales for dangerous rectangles

Each rectangle $\Delta_c(P; s)$ has two arithmetic scales:

- The denominator q of its rational center.
- A height $H(P; s)$ that governs its concentration near a rational line.

The bounds needed for the overall construction are

$$q \leq H(P; s) \leq q^2.$$

We group rectangles by comparable denominators and comparable heights.

The attached-line definition of height is deferred to a backup slide.

The multi-weight simplex lemma: Lemma 3.4

Consider dangerous rectangles that

- meet the same small ball B ,
- have comparable denominators and comparable heights,
- satisfy the appropriate smallness conditions.

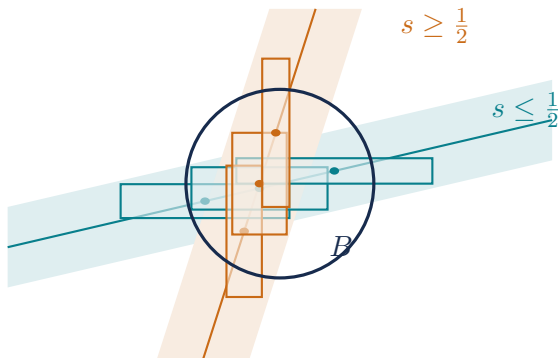
Conclusion

$$\begin{aligned} 0 \leq s \leq \frac{1}{2} & : \text{ one common line neighborhood,} \\ \frac{1}{2} \leq s \leq 1 & : \text{ one common line neighborhood.} \end{aligned}$$

The width is controlled by the largest c/H in the group, up to a scale-dependent constant.

An entire continuum of weights costs at most two neighborhoods per group.

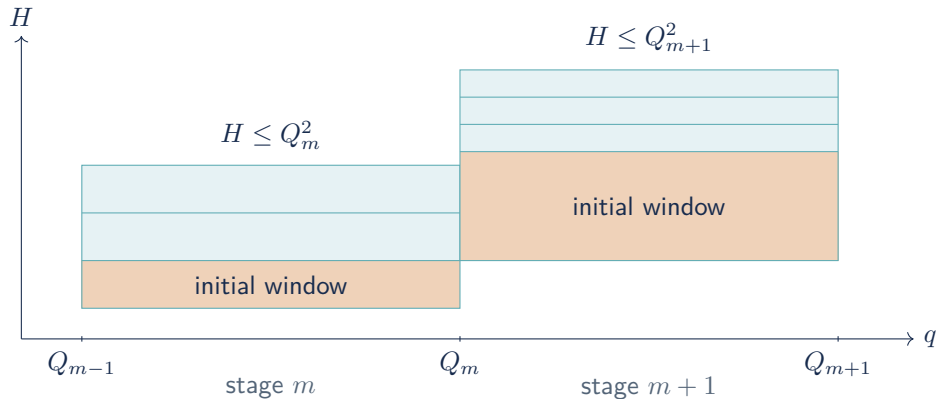
The geometric conclusion



The centers are collinear within each half-family, even though the rectangle shapes vary.

Organizing the dangerous sets

Each denominator block has its own range of heights and height windows.



One move covers a height window across the whole denominator block.

The content of the potential estimates

After clustering, tube radii decay geometrically across denominator ranges. Their costs are summable:

$$\sum_i r_i^\gamma \leq (\beta\rho)^\gamma.$$

Proposition 4.1

At the first move of a stage, the initial collection of lower height levels can be covered within one potential budget.

Proposition 4.2

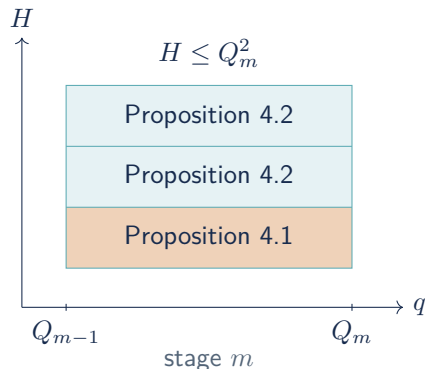
At the appropriate later radius, the collection at the next height level can also be covered within one potential budget.

Each selected height window can be handled in a single Alice move.

The induction: starting the next stage

Suppose stage $m - 1$ has finished.

1. Choose Q_m sufficiently large, using Q_{m-1} and Bob's current radius.
2. Consider the block $Q_{m-1} < q \leq Q_m$.
3. Apply Proposition 4.1 to the initial height window.
4. Apply Proposition 4.2 to later height windows at selected moves.



Process each height level H when Bob's radius first reaches a suitable small multiple of H^{-1} .

Why every stage finishes

For every dangerous set in the current block,

$$H \leq q^2 \leq Q_m^2.$$

- There are only finitely many height levels.
- If $\rho_n \rightarrow 0$, Bob's balls eventually reach every remaining trigger scale.
- Each relevant dangerous set meeting the eventual outcome has therefore been covered during its stage.

The next stage handles the next denominator block.

Induction covers all denominators.

The outcome of the game

Assume Bob's balls shrink to $z_\infty = (x, y)$.

Case 1: the point lies in one of Alice's marked tubes

Alice wins by the rule of the potential game.

Case 2: the point avoids every marked tube

It avoids all the relevant dangerous sets. Corollary 3.2 gives

$$Q_m \min_{1 \leq q \leq Q_m} \langle qx \rangle \langle qy \rangle \geq \eta^2 \quad \text{for every } m.$$

Since $Q_m \rightarrow \infty$, we have $z_\infty \in E$.

The strategy is potential winning for every β, γ , hence E is HAW.

The ambient proof: what we have established

1. Small-product regions admit a cover by rectangles of all weights and coordinate strips.
2. Arithmetic clustering puts each scale group inside only a few line neighborhoods.
3. Potential estimates make the covers legal at moves selected by Bob's shrinking radii.

Counterexamples to uniform Littlewood are HAW in $\mathbb{R}^2$.

Prescribing the stage endpoints

The same construction works with prescribed scales satisfying

$$\boxed{\frac{Q_{m+1}}{Q_m^3} \longrightarrow +\infty.} \quad (6.2)$$

The cubic-growth case

After discarding finitely many terms, Alice can use these Q_m as the stage endpoints. In a shrinking play, every outcome avoiding her tubes satisfies

$$Q_m \min_{1 \leq q \leq Q_m} \langle qx \rangle \langle qy \rangle \geq \eta^2 \quad \text{for all sufficiently large } m.$$

Thus E_Q is HAW.

Changing finitely many scales does not change E_Q .

Every power greater than one gives cubic subsequences

Suppose $Q_{m+1} \geq aQ_m^\tau$ with $a > 0$ and $\tau > 1$. Choose $k \geq 3$ with $\tau^k > 3$.

$$\frac{Q_{m+k}}{Q_m^3} \geq a^{1+\tau+\dots+\tau^{k-1}} Q_m^{\tau^k-3} \longrightarrow +\infty.$$

Split the sequence by the index modulo k :

$$\begin{array}{ccccccc} r = 0 & \text{---} & Q_0 & \text{---} & Q_k & \text{---} & Q_{2k} \text{ ---} \longrightarrow \dots \\ r = 1 & \text{---} & Q_1 & \text{---} & Q_{k+1} & \text{---} & Q_{2k+1} \text{ ---} \longrightarrow \dots \\ & & & & \vdots & & \\ r = k-1 & \text{---} & Q_{k-1} & \text{---} & Q_{2k-1} & \text{---} & Q_{3k-1} \text{ ---} \longrightarrow \dots \end{array}$$

$$\mathbf{Q}^{(r)} = (Q_{r+kj})_{j \geq 0}, \quad 0 \leq r < k.$$

Each $\mathbf{Q}^{(r)}$ satisfies the cubic-growth condition.

Reassembling the sequence: Theorem 1.6

The cubic-growth case gives HAW for every $E_{\mathbf{Q}^{(r)}}$.

$$E_{\mathbf{Q}} = \bigcap_{r=0}^{k-1} E_{\mathbf{Q}^{(r)}}.$$

Indeed: for a point in the intersection, each subsequence has an eventual positive lower bound η_r^2 . There are only finitely many:

$$\min_{0 \leq r < k} \eta_r^2 > 0.$$

This lower bound works at every sufficiently late scale of the original sequence.

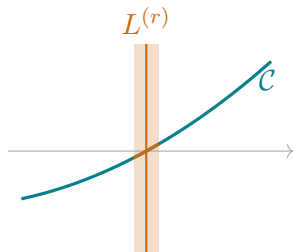
Countable-intersection stability

A finite intersection of HAW sets is HAW. Therefore $E_{\mathbf{Q}}$ is HAW for every $\tau > 1$ in Theorem 1.6.

Proof on curves: transverse and tangent intersections

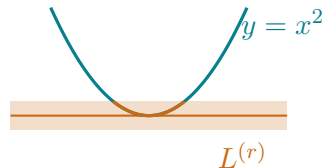
Ambient HAW does not by itself imply winning on a curve.

Transverse intersection



parameter length $O(r)$

Tangential intersection



$$|y| \leq r \implies |x| \leq \sqrt{r}$$

We need the arithmetic information in Alice's tubes.

The arithmetic information carried by each tube

On a coordinate arc write $\phi = (X, Y) : \mathbb{R} \rightarrow \mathcal{C}$. For a primitive integral line put

$$L : Ax + By + C = 0, \quad \xi = \max\{|A|, |B|\}, \quad F(t) = AX(t) + BY(t) + C.$$

For cubic-growth sequences, the ambient strategy can ensure

$$\boxed{r \xi |AB| < \varepsilon} \quad \text{with any prescribed } \varepsilon > 0.$$

Fix $c > 0$ and define

$$D_c = \{t : \xi^2 |F(t)| \geq c \text{ for every primitive integral line } L\}.$$

Geometrically, $\phi(D_c)$ avoids the open tube around each L of radius

$$\frac{c}{\xi^2 \sqrt{A^2 + B^2}} \asymp \frac{c}{\xi^3}.$$

Proof on curves: the local goal

Assume again that the prescribed sequence satisfies

$$\frac{Q_{m+1}}{Q_m^3} \longrightarrow +\infty.$$

Fix $c > 0$. Work on a bounded interval J where

$$m \leq |X'(t)|, |Y'(t)| \leq M, \quad m > 0.$$

Local goal: $S_{\mathbf{Q},c}$ is HAW on J

$$S_{\mathbf{Q},c} := \phi^{-1}(E_{\mathbf{Q}}) \cup (\mathbb{R} \setminus D_c)$$

For outcomes in D_c , we must force their images into $E_{\mathbf{Q}}$. Other outcomes already belong to the target.

Transverse tubes give short parameter intervals

Case 1: throughout B_n ,

$$|F'(t)| \geq \frac{m\xi}{2}.$$

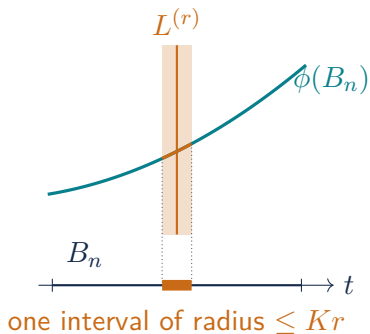
Inside the tube,

$$|F(t)| \leq \sqrt{2} \xi r.$$

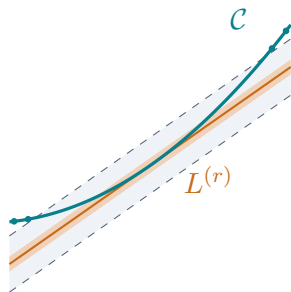
The mean-value theorem gives

$$\text{diam}\{t \in B_n : \phi(t) \in L^{(r)}\} \leq \frac{4\sqrt{2}}{m} r.$$

Thus Alice needs one interval of radius at most Kr , where $K = 2\sqrt{2}/m$.



Nearly tangent tubes miss the relevant Diophantine points



wider tube excluded by D_c
radius $\asymp c/\xi^3$

Case 2: for some $u \in B_n$,

$$|F'(u)| = |AX'(u) + BY'(u)| < m\xi/2.$$

Cancellation forces

$$|AB| > \frac{m}{2M}\xi^2.$$

Hence every $t \in J$ with $\phi(t) \in L^{(r)}$ satisfies

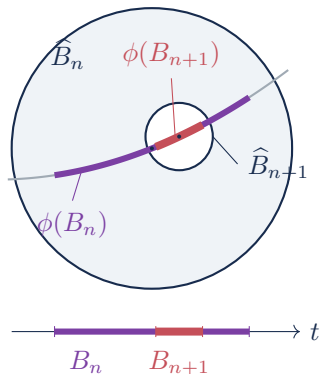
$$\xi^2|F(t)| \leq \sqrt{2}\xi^3r < \frac{2\sqrt{2}M}{m}\varepsilon < c,$$

for sufficiently small ε .

But $t \in D_c$ requires $\xi^2|F(t)| \geq c$.

This tube contains no point of $\phi(D_c \cap J)$, so Alice can ignore it.

From Bob's parameter intervals to Bob's plane balls



Bob plays $B_n = B(t_n, \rho_n) \subset J$. Since ϕ is $\sqrt{2}M$ -Lipschitz, put

$$\hat{B}_n = B(\phi(t_n), \sqrt{2}M\rho_n).$$

These plane balls are nested and contain $\phi(B_n)$.

Run the structured ambient strategy on $\hat{B}_n$.

- Replace transverse tubes by their short parameter intervals.
- Ignore nearly tangent tubes.

The potential budget and the conditional target

If a transverse tube has radius $r_{n,i}$, its parameter interval has radius $\varrho_{n,i} \leq Kr_{n,i}$, with $K = 2\sqrt{2}/m$.

$$\begin{aligned}\sum_i \varrho_{n,i}^\gamma &\leq K^\gamma \sum_i r_{n,i}^\gamma \\ &\leq (K\tilde{\beta}\sqrt{2}M\rho_n)^\gamma \leq (\beta\rho_n)^\gamma.\end{aligned}$$

Choose the ambient parameter $\tilde{\beta}$ small enough.

What this strategy guarantees

An outcome $t_\infty \in D_c$ avoiding Alice's intervals maps to a point avoiding every ambient tube, hence to $E_{\mathbf{Q}}$.

Therefore, for each $c > 0$, the conditional target

$$S_{\mathbf{Q},c} := \phi^{-1}(E_{\mathbf{Q}}) \cup (\mathbb{R} \setminus D_c)$$

is HAW locally on J .

Badly approximable points and the sets D_c

$$\mathbf{Bad}\left(\frac{1}{2}, \frac{1}{2}\right) = \left\{ (x, y) : \inf_{\substack{(A,B) \in \mathbb{Z}^2 \\ (A,B) \neq (0,0)}} \xi^2 \langle Ax + By \rangle > 0 \right\}, \quad \xi = \max\{|A|, |B|\}.$$

A fixed lower bound on the curve

$$D_c = \left\{ t : \xi^2 |AX(t) + BY(t) + C| \geq c \text{ for every primitive integral line } L \right\}.$$

Here $L : Ax + By + C = 0$. Since $\langle AX(t) + BY(t) \rangle = \min_{C \in \mathbb{Z}} |AX(t) + BY(t) + C|$,

$$D := \bigcup_{n \geq 1} D_{1/n} = \phi^{-1}\left(\mathbf{Bad}\left(\frac{1}{2}, \frac{1}{2}\right)\right).$$

Badly approximable points remove the unknown constant

Recall $D = \bigcup_{n \geq 1} D_{1/n} = \phi^{-1}(\mathbf{Bad}(\frac{1}{2}, \frac{1}{2}))$.

On a nondegenerate local arc, **An–Beresnevich–Velani (2018)** together with **Badziahin–Harrap–Nesharim–Simmons (2025)** imply that D is HAW.

Each conditional target $S_{\mathbf{Q}, 1/n}$ is also HAW there.

$$D \cap \bigcap_{n \geq 1} S_{\mathbf{Q}, 1/n} = D \cap \phi^{-1}(E_{\mathbf{Q}}).$$

Indeed, if $t \in D$, then $t \in D_{1/n}$ for some n . Membership in $S_{\mathbf{Q}, 1/n}$ therefore forces $\phi(t) \in E_{\mathbf{Q}}$.

Countable-intersection stability makes the displayed set HAW. Hence $\phi^{-1}(E_{\mathbf{Q}})$ is HAW locally.

Patching the curve and restoring every power $\tau > 1$

Write

$$\Delta_\phi = X'Y'' - Y'X'', \quad U = \{t : \Delta_\phi(t)X'(t)Y'(t) \neq 0\}.$$

1. On $\{\Delta_\phi \neq 0\}$, zeros of X' or Y' are isolated. The curvature hypothesis therefore makes U open and HAW.
2. Cover U by the local intervals used above. Local HAW on this HAW open set gives HAW on the whole parameter line.
3. For $Q_{m+1} \geq aQ_m^\tau$, split into the same finitely many cubic-growth subsequences and intersect.

$E_Q \cap \mathcal{C} \text{ is HAW on } \mathcal{C}.$

Since $E_Q \subseteq E$, the same holds for $E \cap \mathcal{C}$.

Local-to-global principle: Lemma 2.3 of the paper. HAW is invariant under C^1 changes of coordinates.

The arithmetic height

An's attached-line lemma associates to $P = (p/q, r/q)$ and $s \in [0, 1]$ a line

$$L(P; s) : Ax + By + C = 0, \quad A, B, C \in \mathbb{Z},$$

through P , with $(A, B) \neq (0, 0)$ and

$$|A| \leq q^s, \quad |B| \leq q^{1-s}.$$

Set

$$H(P; s) = q \max\{|A|, |B|\}.$$

Then $q \leq H(P; s) \leq q^2$, and

$$\Delta_c(P; s) \subseteq L(P; s)^{(2c/H(P; s))}.$$

An, Duke Math. J. 165 (2016), Lemma 3.1. This is Lemma 3.3 of our paper.

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