

UChicago Dynamics Seminar

Full Hausdorff dimension of the Folklore set

Vasiliy Neckrasov

Joint work with

Reynold Fregoli, Taehyeong Kim,
Dmitry Kleinbock and Anurag Rao

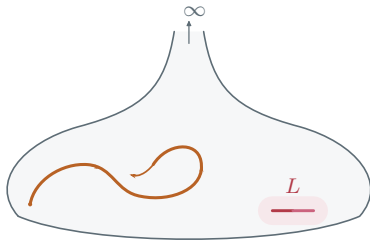
October 26, 2026

The dynamical setting

- G : a connected Lie group with discrete center
- $\Gamma \subset G$: a lattice
- $X = G/\Gamma$: a noncompact homogeneous space
- μ_X : the Haar probability measure on X
- $(g_t)_{t \in \mathbb{R}} \subset G$: an Ad-diagonalizable one-parameter group
Its action on X is mixing, hence ergodic.
- $F = \{g_t : t \geq 0\}, \quad F_- = \{g_t : t \leq 0\}$

Uniformly staying away from a compact set

How large is the set of points whose trajectories eventually uniformly stay away from a very small compact set L ?



Proposition

Let $L \subset X$ be nonempty and compact. Almost every forward trajectory has accumulation points in L . In particular,

$$\mu_X(\{x \in X : \exists \rho > 0, T \geq 0, \text{dist}_X(g_t x, L) \geq \rho \ \forall t \geq T\}) = 0.$$

This follows from ergodicity of the g_t -action.

Can this set nevertheless have full Hausdorff dimension?

Smallness adapted to F

In order to obtain a large set of trajectories not accumulating anywhere in L , we need L to be small in a certain sense.

We want to define a notion describing the smallness of L with respect to the flow.

Definition

A compact set $L \subset X$ is $F_{\pm}$ -small if there exist compact sets $L_+, L_- \subset X$ such that

$$L = L_+ \cup L_-,$$

$$\mu_X(\overline{FL_+} \cup \overline{F_-L_-}) = 0.$$

Eventually escaping: the notation

Let $X^* = X \cup \{\infty\}$ be the one-point compactification.

For $Y \subseteq X^*$, define

$$E(F, Y) = \left\{ x \in X : \begin{array}{l} \text{for every } y \in Y \text{ there is } T \geq 0 \\ \text{such that } y \notin \overline{\{g_tx : t \geq T\}} \end{array} \right\}.$$

Closure is taken in X^* .

For compact L , this means eventually staying uniformly away from L :

$$\exists \rho > 0, T \geq 0 : \quad \text{dist}(g_tx, L) \geq \rho \quad \forall t \geq T.$$

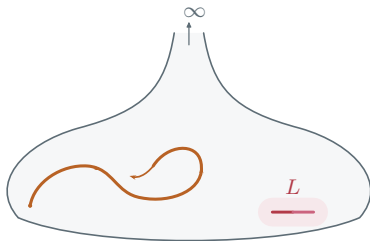
Theorem 0: escaping a small compact set

Definition

A subset of X is thick if it has full Hausdorff dimension in every nonempty open subset of X .

Theorem 0 (Kleinbock–Margulis, '96)

Let L be $F_{\pm}$ -small. Then $E(F, L)$ is thick in X .



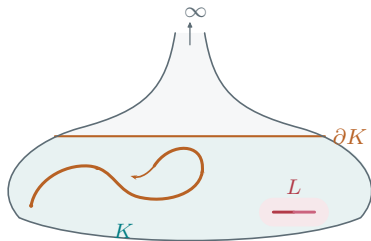
Theorem 0: bounded trajectories suffice

Definition

A subset of X is thick if it has full Hausdorff dimension in every nonempty open subset of X .

Theorem 0 (Kleinbock–Margulis, '96)

Let L be $F_{\pm}$ -small. Then $E(F, L) \cap E(F, \infty)$ is thick in X .



$E(F, \infty)$ is the set of points with bounded trajectories.

Both sets have measure zero. Bounded trajectories have full Hausdorff dimension.

Replacing bounded by unbounded

What if we consider unbounded trajectories instead?

$$A(F, Y) := X \setminus E(F, Y) = \left\{ x \in X : \exists t_j \rightarrow \infty \text{ with } \lim_{j \rightarrow \infty} g_{t_j} x \in Y \right\}.$$

$A(F, \infty)$ is the set of points with unbounded trajectories.

Note: $\mu_X(E(F, \infty)) = 0$, $\mu_X(A(F, \infty)) = 1$.

Can we prove that $E(F, L) \cap A(F, \infty)$ also has full Hausdorff dimension?

Full measure does not settle the intersection

$$\mu_X(E(F, L)) = 0, \quad \mu_X(A(F, \infty)) = 1.$$

Measure alone gives no answer about $E(F, L) \cap A(F, \infty)$.

In some settings, bounded trajectories form a winning set.

Winning properties are useful for intersections.

The complement, unbounded trajectories, is not winning for the same game.

So the new question needs a different argument.

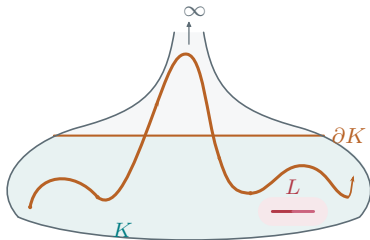
Theorem 1: a conditional answer

Theorem 1

Let L be $F_{\pm}$ -small, with FL_+ and F_-L_- closed.

Suppose there exist $y \in E(F, L) \cap A(F, \infty)$, a compact set $C \subset X$, and $t_j \rightarrow \infty$ such that $g_{t_j}y \in C$ for every j .

Then $E(F, L) \cap A(F, \infty)$ is thick in X .



Theorem 1: a conditional answer

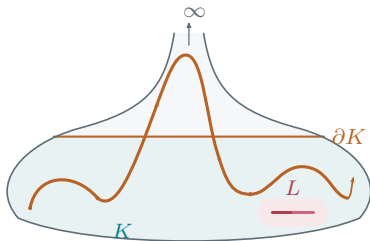
$A(F, X)$: trajectories returning to a fixed compact set at arbitrarily large times.

Theorem 1

Let L be $F_{\pm}$ -small, with FL_+ and F_-L_- closed.

Suppose $E(F, L) \cap A(F, \infty) \cap A(F, X) \neq \emptyset$.

Then $E(F, L) \cap A(F, \infty) \cap A(F, X)$ is thick in X .



Given one strangely behaving trajectory, we can print out many trajectories copying its behavior.

Existence of even one such trajectory is a separate problem.

The space of lattices

$$X_d = \mathrm{SL}_d(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z})$$

Unimodular lattices in $\mathbb{R}^d$, where $d = m + n > 2$ and $m, n \geq 1$

$$g_t = \mathrm{diag}(e^{\alpha_1 t}, \dots, e^{\alpha_m t}, e^{-\beta_1 t}, \dots, e^{-\beta_n t})$$

$$\alpha_i, \beta_j > 0, \quad \sum_i \alpha_i = \sum_j \beta_j = 1$$

The standard choice: $\alpha_i = 1/m$, $\beta_j = 1/n$

$$g_t = \begin{pmatrix} e^{t/m} I_m & 0 \\ 0 & e^{-t/n} I_n \end{pmatrix}$$

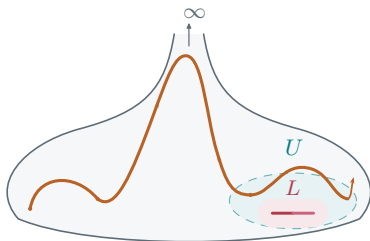
Theorem 2: existence near the prescribed set

Theorem 2

Let $d = m + n > 2$ and let F be the positive weighted lattice flow defined above.

For every nonempty compact $L \subset X_d$ and every open set $U \supset L$,

$$E(F, L) \cap A(F, \infty) \cap A(F, U) \neq \emptyset.$$



Remark: No smallness assumption on L here.

The accumulation point can lie in any chosen open neighborhood of L .

Note: Our proof of Theorem 2 follows the recent method of Agin and Weiss. Khintchine and then Jarník used this approach to construct singular systems in Diophantine approximation.

Reduction to expanding fibers

$$\Phi_t(g) = g_t g g_{-t}.$$

$$H = \{h \in G : \Phi_{-t}(h) \rightarrow e \text{ as } t \rightarrow \infty\}.$$

$$H_s = \left\{z \in G : \overline{\{\Phi_t(z) : t \geq 0\}} \text{ is compact} \right\}_0.$$

The subscript 0 means the identity component. H_s includes neutral directions.

Choose small open neighborhoods $U_H \subset H$, $U_s \subset H_s$ so that $(h, z) \mapsto hz$ gives local coordinates on G .

Assume $E(F, L) \cap A(F, \infty) \cap A(F, X)$ is thick on every expanding leaf.

For all $x_0 \in X$ and $z \in U_s$,

$$\dim_H \{h \in U_H : hzx_0 \in E(F, L) \cap A(F, \infty) \cap A(F, X)\} = \dim H.$$

By fiber slicing in the local product chart,

$$\begin{aligned} \dim_H \{hz : h \in U_H, z \in U_s, hzx_0 \in E(F, L) \cap A(F, \infty) \cap A(F, X)\} \\ = \dim H + \dim H_s = \dim G = \dim X. \end{aligned}$$

An expanding matrix slice

$$H_0 = \left\{ u_\Theta = \begin{pmatrix} I_m & \Theta \\ 0 & I_n \end{pmatrix} : \Theta \in \text{Mat}_{m,n}(\mathbb{R}) \right\}$$

$H_0 \subseteq H$ for the weighted diagonal flow.

$$\Phi_t(u_\Theta) = u_{\Theta(t)}, \quad \Theta(t)_{ij} = e^{(\alpha_i + \beta_j)t} \Theta_{ij}$$

$\alpha_i + \beta_j > 0$: every matrix coordinate expands.

For the standard flow, $\alpha_i = 1/m$, $\beta_j = 1/n$, and $H_0 = H$.

$$x_\Theta = u_\Theta \mathbb{Z}^d$$

For existence, finding one such point is enough.

Statements regarding points in H

Theorem 1'

Under the hypotheses of Theorem 1, suppose $E(F, L) \cap A(F, \infty) \cap A(F, X) \neq \emptyset$.
Then, for every $x_0 \in X$,

$$\{h \in H : hx_0 \in E(F, L) \cap A(F, \infty) \cap A(F, X)\} \text{ is thick in } H.$$

Theorem 2'

Let F be the positive weighted lattice flow on X_d , with $d = m + n > 2$. For every nonempty compact $L \subset X_d$ and open $U \supset L$, there exists $\Theta \in \text{Mat}_{m,n}(\mathbb{R})$ such that

$$x_\Theta \in E(F, L) \cap A(F, \infty) \cap A(F, U).$$

Moreover, the set of such Θ is dense and uncountable in every nonempty open matrix set.

Diophantine approximation: the standard flow

Let $X = X_d$ and let $\|\cdot\|$ be the supremum norm.

$$g_t = \begin{pmatrix} e^{t/m} I_m & 0 \\ 0 & e^{-t/n} I_n \end{pmatrix}, \quad H = \left\{ u_\Theta = \begin{pmatrix} I_m & \Theta \\ 0 & I_n \end{pmatrix} \right\}$$

$$x_\Theta = u_\Theta \mathbb{Z}^d, \quad \Theta \in \text{Mat}_{m,n}(\mathbb{R})$$

$$g_t u_\Theta \begin{pmatrix} -\mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} e^{t/m}(\Theta \mathbf{q} - \mathbf{p}) \\ e^{-t/n} \mathbf{q} \end{pmatrix}$$

Dirichlet's theorems

$$(*) \quad \begin{cases} \|\Theta \mathbf{q} - \mathbf{p}\|^m < c/T, \\ 0 < \|\mathbf{q}\|^n < T. \end{cases} \quad \begin{array}{l} \mathbf{p} \in \mathbb{Z}^m, \quad \mathbf{q} \in \mathbb{Z}^n \\ \|\cdot\| \text{ is the sup norm.} \end{array}$$

Uniform Dirichlet's theorem

$$\forall \Theta \in \text{Mat}_{m,n}(\mathbb{R}), \forall T \geq 1 :$$

(*) is solvable with $c = 1$ and weak upper bounds.

Asymptotic Dirichlet's theorem

$$\forall \Theta \in \text{Mat}_{m,n}(\mathbb{R}), \forall T_0 \geq 1, \exists T > T_0 :$$

(*) is solvable with $c = 1$.

Dirichlet-improvable matrices

$$(*) \quad \begin{cases} \|\Theta \mathbf{q} - \mathbf{p}\|^m < c/T, \\ 0 < \|\mathbf{q}\|^n < T. \end{cases} \quad \begin{array}{l} \mathbf{p} \in \mathbb{Z}^m, \quad \mathbf{q} \in \mathbb{Z}^n \\ \|\cdot\| \text{ is the sup norm.} \end{array}$$

$$\begin{aligned} \text{DI}_{m,n} = \{ & \Theta \in \text{Mat}_{m,n}(\mathbb{R}) : \\ & \exists c \in (0, 1), \exists T_0 > 1 : \\ & \forall T \geq T_0, (*) \text{ is solvable} \}. \end{aligned}$$

Dani
correspondence
 $\longleftrightarrow$

$$\begin{aligned} \text{DI}_{m,n} = \{ & \Theta \in \text{Mat}_{m,n}(\mathbb{R}) : \\ & x_\Theta \in E(F, L) \}. \end{aligned}$$

Improvement to the uniform
Dirichlet theorem.

$$\begin{aligned} L &= \{\Lambda \in X_d : \lambda_1(\Lambda) = 1\} \\ \lambda_1(\Lambda) &= \min_{\mathbf{v} \in \Lambda \setminus \{0\}} \|\mathbf{v}\| \\ L &= \bigcup_i L_i, \quad L_i = \{\Lambda \in L : e_i \in \Lambda\} \end{aligned}$$

$$\text{Hajós: } L_+ = \bigcup_{i > m} L_i, \quad L_- = \bigcup_{i \leq m} L_i, \quad L = L_+ \cup L_-.$$

L compact; ergodicity and local product give $\text{Leb}(\text{DI}_{m,n}) = 0$.

L is $F_\pm$ -small: $\text{DI}_{m,n}$ is thick, so $\dim_{\text{H}} \text{DI}_{m,n} = mn$.

Singular matrices

$$(*) \quad \begin{cases} \|\Theta \mathbf{q} - \mathbf{p}\|^m < c/T, \\ 0 < \|\mathbf{q}\|^n < T. \end{cases} \quad \begin{array}{l} \mathbf{p} \in \mathbb{Z}^m, \quad \mathbf{q} \in \mathbb{Z}^n \\ \|\cdot\| \text{ is the sup norm.} \end{array}$$

$$\begin{aligned} \text{Sing}_{m,n} &= \{ \Theta \in \text{Mat}_{m,n}(\mathbb{R}) : \\ &\quad \forall c > 0, \exists T_0 > 1 : \\ &\quad \forall T \geq T_0, (*) \text{ is solvable} \}. \end{aligned}$$

Dani
correspondence
 $\Longleftrightarrow$

$$\begin{aligned} \text{Sing}_{m,n} &= \{ \Theta \in \text{Mat}_{m,n}(\mathbb{R}) : \\ &\quad x_\Theta \in E(F, X_d) \}. \end{aligned}$$

Arbitrarily strong uniform
Dirichlet improvement.

$$\text{Sing}_{m,n} \subset \text{DI}_{m,n}$$

The trajectory diverges:
it eventually leaves every
compact subset of X_d .

$$\text{Leb}(\text{Sing}_{m,n}) = 0, \quad \dim_{\text{H}} \text{Sing}_{m,n} = mn \left(1 - \frac{1}{m+n}\right) \quad (m+n > 2).$$

$$\text{Sing}_{1,1} = \mathbb{Q}.$$

Dimension: KKLM; DFSU

Badly approximable matrices

$$(*) \quad \begin{cases} \|\Theta \mathbf{q} - \mathbf{p}\|^m < c/T, \\ 0 < \|\mathbf{q}\|^n < T. \end{cases} \quad \begin{array}{l} \mathbf{p} \in \mathbb{Z}^m, \quad \mathbf{q} \in \mathbb{Z}^n \\ \|\cdot\| \text{ is the sup norm.} \end{array}$$

$$\mathrm{BA}_{m,n} = \{ \Theta \in \mathrm{Mat}_{m,n}(\mathbb{R}) :$$

$$\exists c > 0, \exists T_0 > 1 :$$

$$\forall T \geq T_0, (*) \text{ is insoluble} \}.$$

Dani
correspondence
 $\Longleftrightarrow$

$$\mathrm{BA}_{m,n} = \{ \Theta \in \mathrm{Mat}_{m,n}(\mathbb{R}) :$$

$$x_\Theta \in E(F, \infty) \}.$$

The trajectory is bounded:
it stays in a compact
subset of X_d .

$$\mathrm{BA}_{m,n} \subset \mathrm{DI}_{m,n}$$

$$\mathrm{Leb}(\mathrm{BA}_{m,n}) = 0, \quad \dim_{\mathrm{H}} \mathrm{BA}_{m,n} = mn.$$

In fact $\mathrm{BA}_{m,n}$ is thick.

The Folklore set

For $m = n = 1$,

$$\text{DI}_{1,1} = \text{Sing}_{1,1} \cup \text{BA}_{1,1}.$$

For $m + n > 2$, what remains after removing these two sets?

$$\text{FS}_{m,n} := \text{DI}_{m,n} \setminus (\text{Sing}_{m,n} \cup \text{BA}_{m,n}).$$

Dirichlet improvable, non-singular, and not badly approximable.

Is this set nonempty? How large is it?

Towards the Folklore conjecture

Beresnevich–Guan–Marnat–Ramírez–Velani (2022)

$\text{FS}_{m,1}$ is uncountable for every $m \geq 2$.

Hussain–Schleischitz–Ward (2024/25): independent construction

$\text{FS}_{m,n} \neq \emptyset$ if $(m, n) \notin \{(1, 1), (2, 3), (3, 2)\}$;

$\dim_{\text{H}} \text{FS}_{m,n} \geq mn - 2 \max\{m, n\}$.

Das–Fishman–Simmons–Urbański (2024)

$\dim_{\text{H}} \text{FS}_{m,n} \geq mn \left(1 - \frac{1}{m+n}\right) \quad (m + n > 2).$

Folklore conjecture (BGMRV):

$\dim_{\text{H}} \text{FS}_{m,n} = mn \quad (m + n > 2).$

The Folklore conjecture in dynamical language

$$\mathrm{FS}_{m,n} = \{\Theta : x_\Theta \in E(F, L) \cap A(F, \infty) \cap A(F, X_d)\},$$

the matrices whose trajectories are unbounded, non-divergent, and escape the critical locus L .

We now prove the Folklore conjecture.

Theorem 2': for every open $U \supset L$, there is a matrix Θ with $x_\Theta \in E(F, L) \cap A(F, \infty) \cap A(F, U)$.

Corollary: $E(F, L) \cap A(F, \infty) \cap A(F, X_d) \neq \emptyset$.

Theorem 1' makes this intersection thick on every expanding leaf. Since $H \simeq \mathrm{Mat}_{m,n}(\mathbb{R})$, the Folklore set is thick.

$\dim_{\mathrm{H}} \mathrm{FS}_{m,n} = mn$	$(m + n > 2)$
--	---------------

Shadowing theorem

Shadowing theorem (hypotheses)

Let F be mixing and Ad-diagonalizable, and let L be F_+ -small. Put

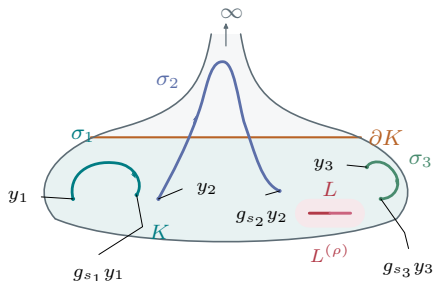
$$Y = \overline{FL_+} \cup \overline{F_-L_-}.$$

Given $y_k \in X$ and $0 < s_k < \infty$, suppose there are a compact $K \subset X$ and $\delta > 0$ such that, for every $k \geq 1$,

$$y_k, g_{s_k} y_k \in K,$$

$$\text{dist}_X(\{y_k, g_{s_k} y_k\}, Y) \geq \delta,$$

$$\text{dist}_X(g_{[0,s_k]}y_k, L) \geq \delta.$$



$$\sigma_i : [0, s_i] \rightarrow X$$

$$\sigma_i(u) = g_u y_i, \quad i = 1, 2, 3, \dots$$

The segments may leave K . Their durations s_k need not be bounded.

Shadowing theorem

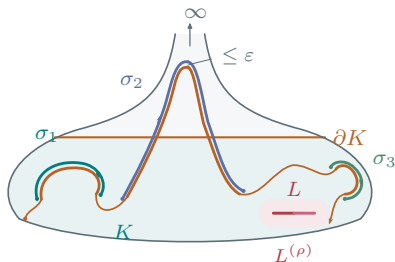
Shadowing theorem (conclusion)

Under these hypotheses, for every $\varepsilon > 0$, there is a thick set of $x \in E(F, L)$ for which there exist times $T_k > 0$ satisfying

$$\text{dist}_X(g_{T_k+u}x, g_u y_k) \leq \varepsilon$$

for every $k \geq 1$ and every $u \in [0, s_k]$.

For every $x_0 \in X$, the corresponding set of $h \in H$, with $x = hx_0$, is thick in H .



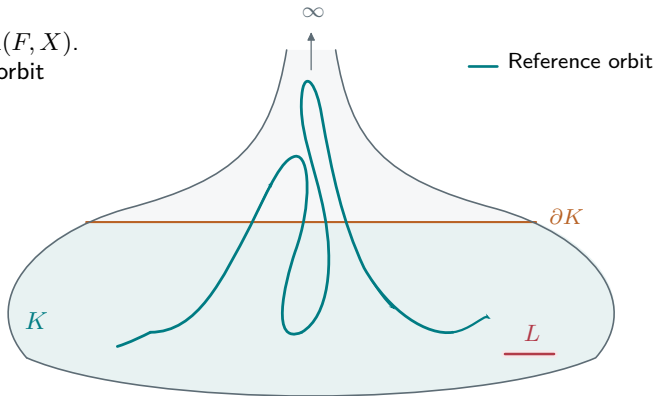
Orange: the constructed trajectory

$$[T_1, T_1 + s_1] \quad [T_2, T_2 + s_2] \quad [T_3, T_3 + s_3]$$

In between the shadowed segments, the motion is not prescribed (but the trajectory stays away from L).

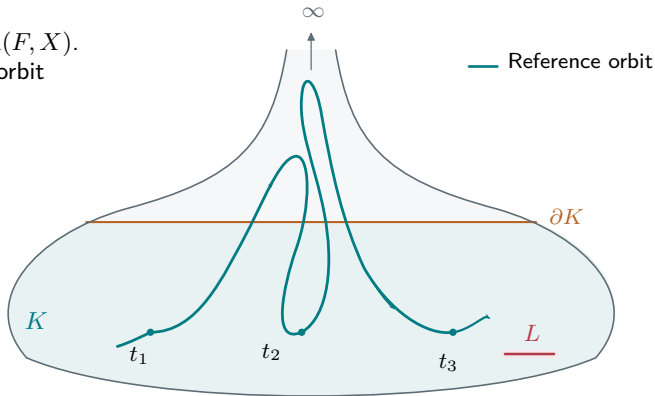
How Shadowing Theorem implies Theorem 1

- Choose $y \in E(F, L) \cap A(F, \infty) \cap A(F, X)$.
Shift past its initial segment so the orbit stays uniformly away from L .



How Shadowing Theorem implies Theorem 1

- Choose $y \in E(F, L) \cap A(F, \infty) \cap A(F, X)$.
Shift past its initial segment so the orbit stays uniformly away from L .
- Its accumulation set avoids the closed set Y . Choose compact K , $\delta > 0$, and returns $g_{t_k} y \in K \setminus Y^{(\delta)}$, $t_k \rightarrow \infty$.

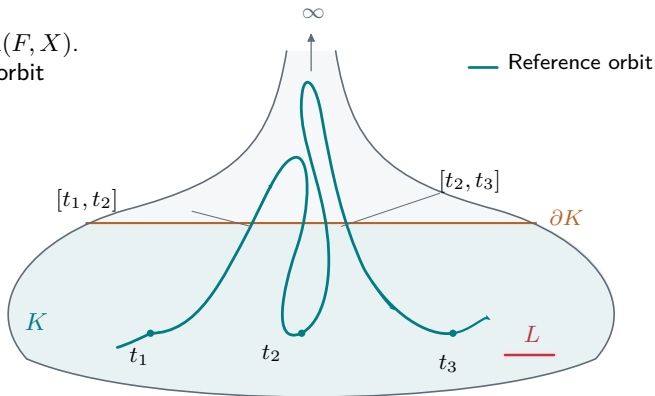


How Shadowing Theorem implies Theorem 1

- Choose $y \in E(F, L) \cap A(F, \infty) \cap A(F, X)$.
Shift past its initial segment so the orbit stays uniformly away from L .

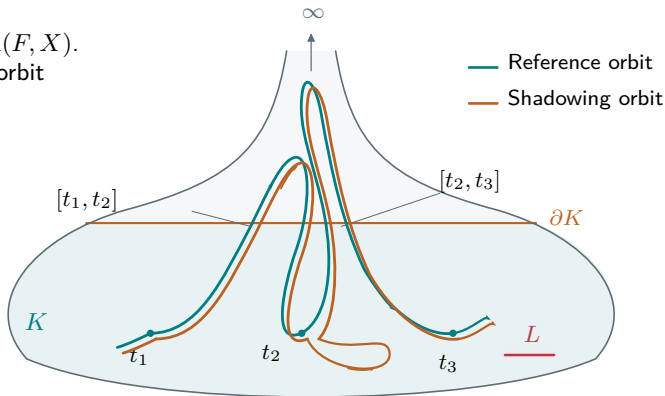
- Its accumulation set avoids the closed set Y . Choose compact K , $\delta > 0$, and returns $g_{t_k} y \in K \setminus Y^{(\delta)}$, $t_k \rightarrow \infty$.

- Put $y_k = g_{t_k} y$ and $s_k = t_{k+1} - t_k$. Choose $s_k \rightarrow \infty$ with increasingly deep excursions between returns.



How Shadowing Theorem implies Theorem 1

- Choose $y \in E(F, L) \cap A(F, \infty) \cap A(F, X)$.
Shift past its initial segment so the orbit stays uniformly away from L .
- Its accumulation set avoids the closed set Y . Choose compact K , $\delta > 0$, and returns $gt_k y \in K \setminus Y^{(\delta)}$, $t_k \rightarrow \infty$.
- Put $y_k = gt_k y$ and $s_k = t_{k+1} - t_k$. Choose $s_k \rightarrow \infty$ with increasingly deep excursions between returns.
- Fixed small-error shadowing gives a thick family in $E(F, L)$. Their trajectories have unbounded excursions and return to a compact neighborhood of K .



Equidistribution of expanding horospherical pieces

We now prove the shadowing theorem.

Let H be the expanding horospherical subgroup of G with respect to g_t :

$$H = \{h \in G : g_{-t}hg_t \xrightarrow[t \rightarrow \infty]{} e\}$$

$$\Phi_t(h) = g_thg_{-t}$$

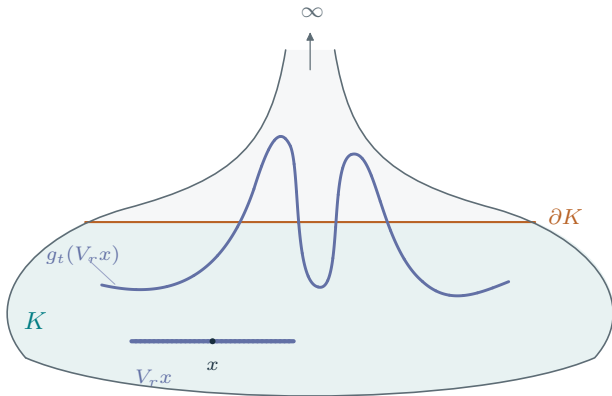
Take a small tile $V_r \subset H$,
whose translates $V_r\gamma$ tile H .

$$g_t(V_rx) = \Phi_t(V_r)g_tx$$

Equidistribution of H with respect to g_t yields:

$$\frac{\mu_H\{h \in V_r : g_thx \in K\}}{\mu_H(V_r)} \longrightarrow \mu_X(K) \quad (t \rightarrow \infty)$$

uniformly for x in compact sets, with fixed K , for which $\mu_X(\partial K) = 0$.



Generic step: whole returning tiles

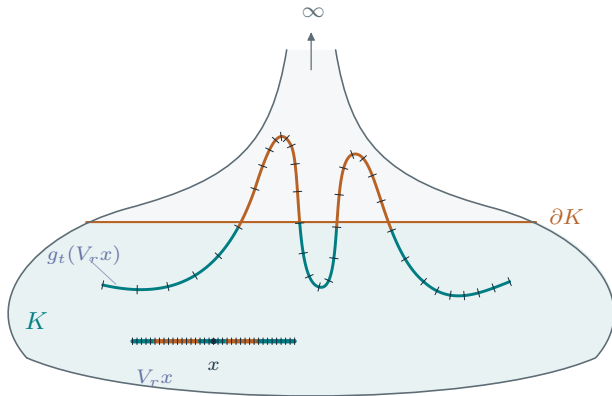
Take a small horospherical tile V_r .

Expand, keep whole returning tiles,
and pull them back.

For r small and t large:

$$W_\gamma = \Phi_{-t}(V_r \gamma) \subset V_r$$

$$g_t W_\gamma x \subset K$$



Teal: whole tiles returning to K

For any $\eta > 0$, choose r small and t large enough so that

$$\#\{\text{returning tiles}\} \geq e^{\chi t}(\mu_X(K) - \eta), \quad \mu_H(\Phi_t V) = e^{\chi t} \mu_H(V).$$

We fix $\eta = \mu_X(K)/2$: retained proportion $\geq \mu_X(K)/2$.

Generic step: avoidance of L

L is $F_{\pm}$ -small; $\mu_X(Y) = 0$.

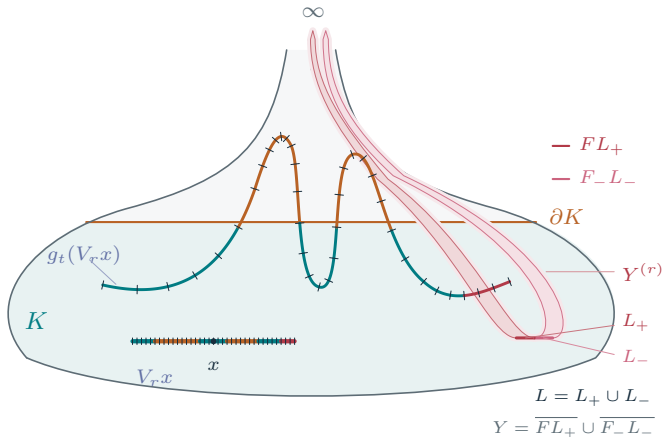
Start from $x \in K \setminus Y^{(r)}$.

Remove also the whole tiles meeting $Y^{(r)}$.

This costs arbitrarily little mass when r is small.

Endpoint lemma: endpoints avoiding $Y^{(r)}$ give an L -avoiding connector:

$$\text{dist}_X(g_{[0,t]}W_\gamma x, L) \geq \iota > 0$$



The same retained proportion $\mu_X(K)/2$ survives.

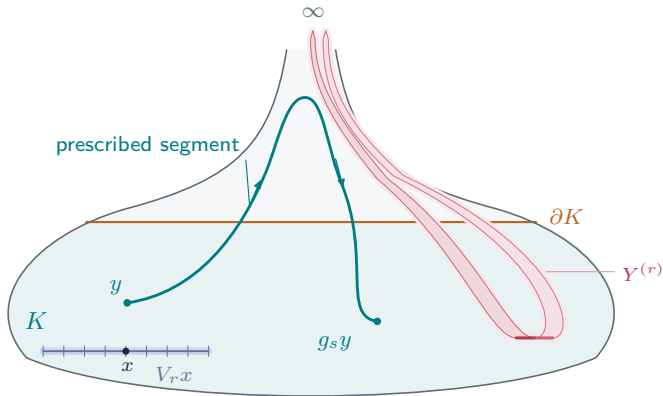
Approach step

K is compact, and L is $F_{\pm}$ -small.

We are given a small horospherical piece $V_r x$.

We want to follow the trajectory segment $g_{[0,s]}y$, given that

$$x, y, g_s y \in K \setminus Y^{(r)}.$$



Note: s can be arbitrarily large, independently of t .

Goal of the approach step

We construct one child

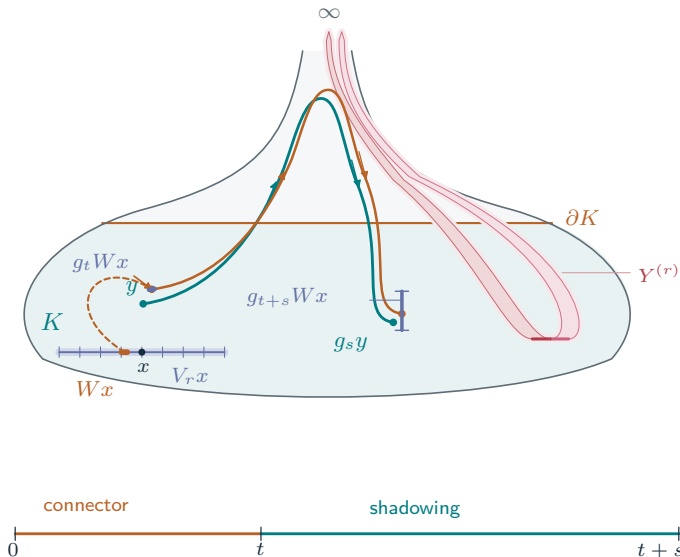
$$W = \Phi_{-(t+s)}(V_r \gamma)$$

such that:

- (i) $W \subset V_r$.
- (ii) The whole child avoids L during the connector $[0, t]$.
- (iii) Every point of the child then follows the trajectory of y within r :

$$\text{dist}_X(g_{t+u}wx, g_u y) \leq r$$

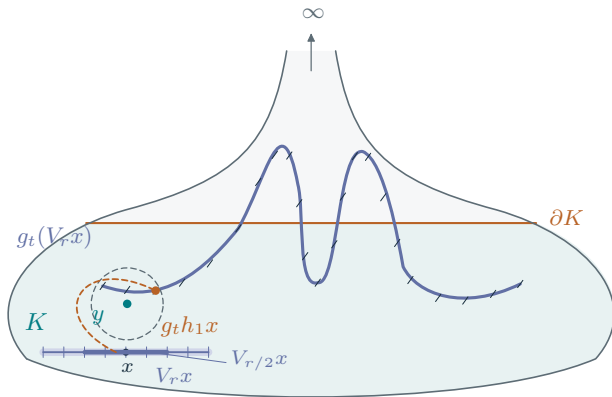
$$(w \in W, 0 \leq u \leq s).$$



Equidistribution gives a point near y

For large t , the expanded horospherical piece meets a small neighborhood of y :
there exists $h_1 \in V_{r/2}$ such that

$$g_t h_1 x \in B_X(y, r/C^2).$$



Note: We use $r/2$ instead of r to later guarantee that the whole neighborhood still lies in V_r .



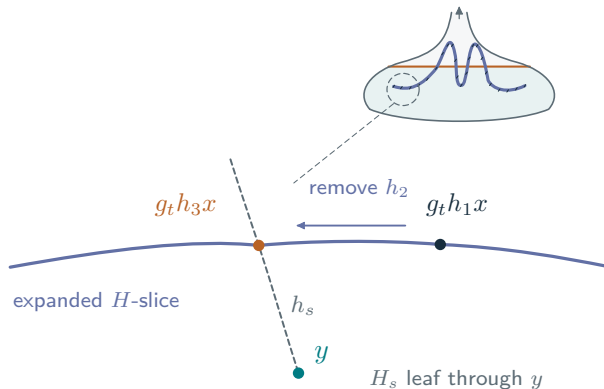
Projection along H_s

The landing error has an H part and an H_s part:

$$g_t h_1 x = h_2 h_s y.$$

The projection of y onto the expanded H -slice along H_s is close to $g_t h_1 x$:

$$g_t h_3 x = h_s y.$$



$h_3 = \Phi_{-t}(h_2^{-1})h_1$, so the correction is tiny when pulled back to the parent.

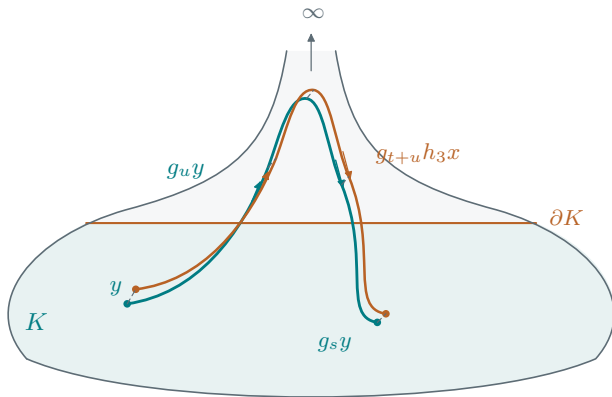
The corrected point follows y for as long as needed

The corrected point differs from y only along H_s .

$\Rightarrow$ This error stays uniformly controlled under the flow:

$$g_{t+u}h_3x = \Phi_u(h_s)g_uy.$$

H_s includes contracting and neutral directions.



The bound on $\Phi_u(h_s)$ is uniform in u . It does not depend on s .



A fine tile for the whole segment

Choose a tile $V_r\gamma$ with $\Phi_{t+s}(h_3) \in \overline{V_r\gamma}$. Then

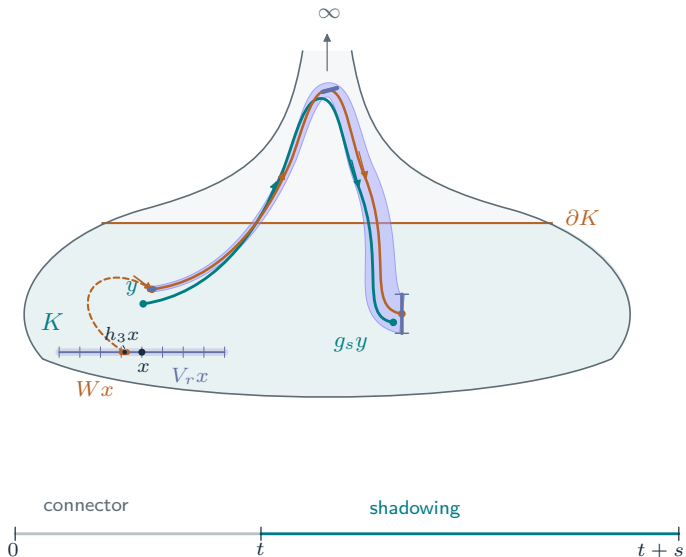
$$W = \Phi_{-(t+s)}(V_r\gamma)$$

is a small tile with $h_3 \in \overline{W}$.

For $0 \leq u \leq s$, we have

$$\Phi_{t+u}(W) = \Phi_{-(s-u)}(V_r\gamma).$$

Each earlier image is obtained by backward contraction of the final tile.



(i) The whole child stays inside the parent

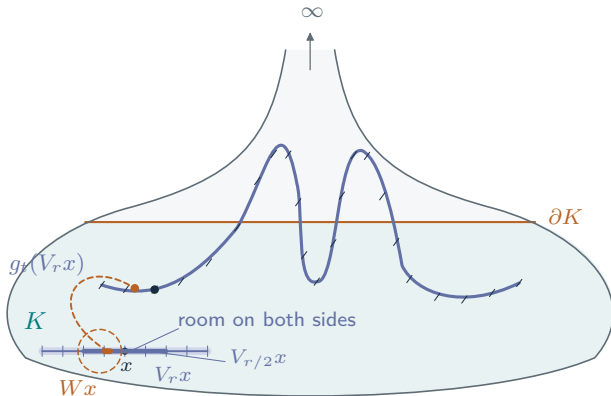
We started with $h_1 \in V_{r/2}$, away from the parent boundary.

For large t , both the correction

$$h_3 h_1^{-1} = \Phi_{-t}(h_2^{-1})$$

and the whole child are small when pulled back, so

$$W \subset V_r.$$



The same lower bound on t works for every duration s .



(ii) Safe endpoints give a safe connector

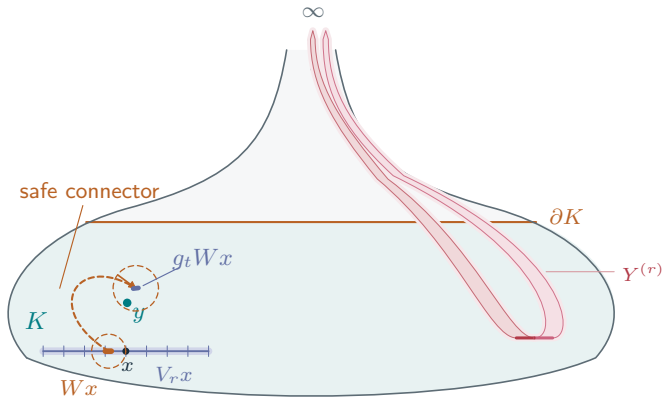
Wx is close to x :

$$\text{dist}_X(Wx, Y) \geq 3r/4.$$

$g_t Wx$ is close to y :

$$\text{dist}_X(g_t Wx, Y) \geq r/4.$$

The endpoint lemma keeps the whole connector away from L .



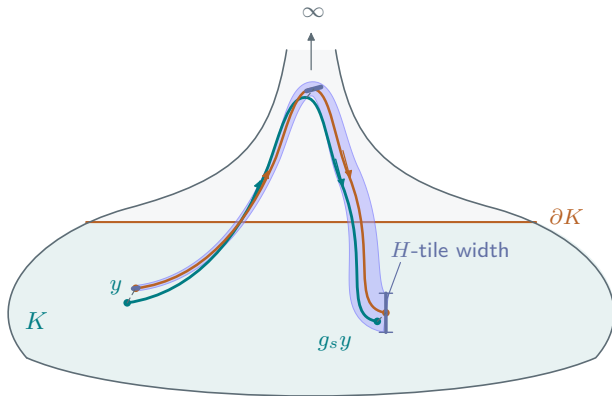
The entire connector stays at least $\iota > 0$ from L , independently of s .



(iii) Every point of the child follows y

Every point of W has two small errors:

- The H_s error stays bounded.
- The H error expands only up to the final tile size.



Overall: if $0 < u < s$, then $\text{dist}_X(g_{t+u}wx, g_u y) \leq r$ for every $w \in W$.



Shadowing proof: two kinds of steps

Fix x_0 . After a finite initial step, start in a small tile $V \subset H$.

$$\mathbb{N} = \mathcal{G} \sqcup \mathcal{A}, \quad \mathcal{A} = \{f(1), f(2), \dots\}$$

$\downarrow$
generic steps

$\nearrow$
approach steps

Generic step

Time t_0



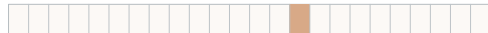
Many good children

Retained proportion $\geq \mu_X(K)/2$

- At the end, every retained tile is back in $K \setminus Y^{(r)}$.
- Both steps preserve avoidance of L .

Approach step $f(n)$

Time $t_0 + s_n$

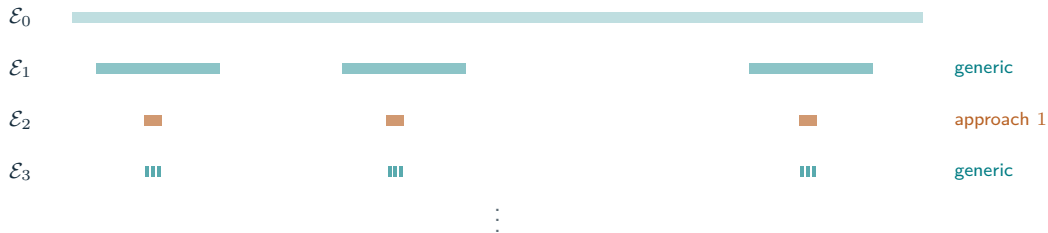


One child follows the n -th segment

Retained proportion $= e^{-\chi(t_0 + s_n)}$

A Cantor-like set of shadowing trajectories

The same construction inside every parent



$$E_\infty = \bigcap_{k \geq 0} \bigcup_{E \in \mathcal{E}_k} E$$

Due to the constraints at every step, we avoid a neighborhood of L .

Due to shadowing at approach steps, we closely follow each desired trajectory during some time interval.

Thus E_∞ is a Cantor-like subset of the desired set in H .

Lower bound for dimension (Urbański)

Δ_j : fraction of mass retained at step j . d_k : largest diameter at level k .

Lemma (Urbański)

$$\dim H - \dim_{\mathrm{H}} E_{\infty} \leq \limsup_{k \rightarrow \infty} \frac{\sum_{j=1}^k \log(1/\Delta_j)}{\log(1/d_{k-1})}$$

Tiles shrink exponentially: $d_k \lesssim e^{-ckt_0}$, $c > 0$.

G_k : number of generic steps through level k .

A_k : total time spent in approach steps through level k .

$$\dim H - \dim_{\mathrm{H}} E_{\infty} \leq \limsup_{k \rightarrow \infty} \frac{\underbrace{G_k \log(2/\mu_X(K))}_{\text{generic steps}} + \underbrace{\chi A_k}_{\text{approach steps}}}{c k t_0}$$

Rare approach steps, full dimension

Choose the approach stages so that their total duration is negligible:

$$A_k = \sum_{f(n) \leq k} (t_0 + s_n) = o(k)$$



Then $G_k/k \rightarrow 1$, so only the generic term remains.

$$\dim H - \dim_{\mathbb{H}} E_{\infty} \leq \frac{\log(2/\mu_X(K))}{c t_0}$$

Take t_0 arbitrarily large.

Repeat inside every open subset of H . This gives thickness.

Thank you for your attention!

References

- D. Kleinbock and G. A. Margulis.
Bounded orbits of nonquasiunipotent flows on homogeneous spaces.
Amer. Math. Soc. Transl. (2) **171** (1996), 141–172.
- M. Einsiedler, D. Kleinbock and A. Rao.
Constructing bounded orbits of special types on homogeneous spaces.
arXiv:2511.16095 (2025).
- A. Agin and B. Weiss.
The Dirichlet spectrum.
arXiv:2412.05858 (2024).

References

- V. Beresnevich, L. Guan, A. Marnat, F. Ramírez and S. Velani.
Dirichlet is not just bad and singular.
Adv. Math. **401** (2022), 108316.
- M. Hussain, J. Schleischitz and B. Ward.
On the Folklore set and Dirichlet spectrum for matrices.
arXiv:2402.13451 (2024; revised 2025).
- T. Das, L. Fishman, D. Simmons and M. Urbański.
A variational principle in the parametric geometry of numbers.
Adv. Math. **437** (2024), 109435.

References

- S. G. Dani.
Divergent trajectories of flows on homogeneous spaces and Diophantine approximation.
J. Reine Angew. Math. **359** (1985), 55–89.

- S. Kadyrov, D. Kleinbock, E. Lindenstrauss and G. A. Margulis.
Singular systems of linear forms and non-escape of mass in the space of lattices.
J. Anal. Math. **133** (2017), 253–277.

- M. Urbański.
The Hausdorff dimension of the set of points with nondense orbit under a hyperbolic dynamical system.
Nonlinearity **4** (1991), 385–397.