

UMaryland Dynamics Seminar

Bounded trajectories on homogeneous spaces for subsets without semigroup structure

Vasiliy Neckrasov

Joint work with Dmitry Kleinbock

The homogeneous setup

$$X = G/\Gamma, \quad g_t = \exp(t\mathbf{v}), \quad F = \{g_t : t \geq 0\}.$$

- G is connected semisimple, without compact factors.
- Γ is an irreducible lattice.
- $\text{ad } \mathbf{v}$ is diagonalizable over $\mathbb{R}$, and $\mathbf{v} \neq 0$.

Normalize Haar measure: $\mu_X(X) = 1$. These hypotheses imply that (g_t) is **mixing**, hence ergodic.

The **expanding horospherical subgroup** of $\mathbf{v}$ is

$$U = U_+(g_1) := \{u \in G : g_{-t}ug_t \xrightarrow[t \rightarrow +\infty]{} e\}.$$

Birkhoff theorem and bounded trajectories

Theorem (Birkhoff)

For every $f \in L^1(X, \mu_X)$, ergodicity gives

$$\frac{1}{T} \int_0^T f(g_t x) dt \longrightarrow \int_X f d\mu_X \quad \text{for } \mu_X\text{-almost every } x.$$

Define the set of bounded trajectories by

$$\mathcal{B}(F) := \{x \in X : Fx \text{ is relatively compact}\}.$$

Corollary

If X is noncompact, then $\mu_X(\mathcal{B}(F)) = 0$.

Zero measure

Fix *any* $x \in X$. Let μ_U be Haar measure on U .

$$\mu_U\{u \in U : ux \in \mathcal{B}(F)\} = 0.$$

Why does the ambient statement imply this?

Put $U^- = U_+(g_{-1})$. The multiplication map

$$U^- \times Z_G(F)^0 \times U \longrightarrow G, \quad (s, c, u) \longmapsto scu$$

near (e, e, e) is a **bi-Lipschitz diffeomorphism** onto an open subset of G .

Boundedness survives multiplication in the first two directions. A positive-measure set in a U -leaf would therefore give positive ambient measure.

How large can this set of measure zero be?

Theorem (Kleinbock–Margulis, 1996)

For every $x \in X$ and every nonempty open $V \subset U$,

$$\dim_H \{u \in V : ux \in \mathcal{B}(F)\} = \dim U.$$

The set is **thick in U** : it has full Hausdorff dimension locally.

Consequently, for every nonempty open $W \subset X$,

$$\dim_H (W \cap \mathcal{B}(F)) = \dim X.$$

The compact set containing the trajectory may depend on the starting point.

Kleinbock–Margulis (1996), **Bounded orbits of nonquasiunipotent flows**.

Diophantine approximation

Let $M_{m,n}$ denote the real $m \times n$ matrices, and let $\Theta \in M_{m,n}$.

$$|\cdot| = \text{supremum norm}, \quad \|\mathbf{y}\| := \min_{\mathbf{p} \in \mathbb{Z}^k} |\mathbf{y} - \mathbf{p}| \quad \text{for } \mathbf{y} \in \mathbb{R}^k.$$

Theorem (Dirichlet)

There are infinitely many $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\|\Theta \mathbf{q}\|^m \leq \frac{1}{|\mathbf{q}|^n}.$$

This gives approximations for an unbounded sequence of times $T = |\mathbf{q}|^n$.

This is **asymptotic approximability** at the rate $f_1(T) = 1/T$.

Badly approximable matrices

How much can the Dirichlet rate improve?

A matrix Θ is badly approximable if, for some $c > 0$, the system

$$\|\Theta \mathbf{q}\|^m \leq \frac{c}{T}, \quad |\mathbf{q}|^n \leq T, \quad \mathbf{q} \in \mathbb{Z}^n \setminus \{0\},$$

has no solutions for all sufficiently large T .

Equivalently,

$$\Theta \in \mathbf{BA}_{m,n} \iff \inf_{\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}} |\mathbf{q}|^n \|\Theta \mathbf{q}\|^m > 0.$$

The space of lattices

Put $d = m + n$. Let

$$X_d = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}) = \{\text{unimodular lattices in } \mathbb{R}^d\}.$$

$$u_\Theta = \begin{pmatrix} 1_m & \Theta \\ 0 & 1_n \end{pmatrix}, \quad g_t = \mathrm{diag}(e^{t/m} 1_m, e^{-t/n} 1_n).$$

$$U = \{u_\Theta : \Theta \in M_{m,n}\} = U_+(g_1).$$

This U is an abelian, minimal nontrivial horospherical subgroup.

Its conjugation law makes expansion explicit:

$$g_t u_\Theta g_{-t} = u_{e^{t(1/m+1/n)}\Theta}.$$

Dani correspondence for bad approximability

The integer pair $(\mathbf{p}, \mathbf{q})$ gives the lattice vector

$$g_t u_\Theta \begin{pmatrix} -\mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} e^{t/m}(\Theta \mathbf{q} - \mathbf{p}) \\ e^{-t/n} \mathbf{q} \end{pmatrix}.$$

Mahler's compactness criterion

A family in X_d is relatively compact if and only if all its nonzero lattice vectors have a common positive lower bound in norm.

Therefore,

$$\Theta \in \mathbf{BA}_{m,n} \iff \{g_t u_\Theta \mathbb{Z}^d : t \geq 0\} \text{ is bounded in } X_d.$$

Zero measure, full Hausdorff dimension

The map $\Theta \mapsto u_\Theta$ identifies $M_{m,n}$ with U .

Dani correspondence + Birkhoff theorem + Kleinbock–Margulis

$$\text{Leb}(\mathbf{BA}_{m,n}) = 0, \quad \dim_H(\mathbf{BA}_{m,n} \cap V) = mn$$

for every nonempty open $V \subset M_{m,n}$.

Thus badly approximable matrices form a **thick set of measure zero**.

The zero-measure result already follows from the classical Khintchine–Groshev theorem. Schmidt (1969) proved the winning property, which implies full Hausdorff dimension.

Khintchine (1924, 1926); Groshev (1938). Schmidt (1969), Badly approximable systems of linear forms.

Weight functions in Diophantine approximation

Replace fixed powers by increasing bijections

$$\alpha_i : (0, 1] \longrightarrow (0, 1], \quad \beta_j : [1, \infty) \longrightarrow [1, \infty).$$

Write $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\vec{\beta} = (\beta_1, \dots, \beta_n)$.

Normalization

$$\prod_{i=1}^m \alpha_i(1/T) = \frac{1}{T}, \quad \prod_{j=1}^n \beta_j(T) = T.$$

The classical case is $\alpha_i(s) = s^{1/m}$, $\beta_j(T) = T^{1/n}$.

Dirichlet theorem with weight functions

Theorem (Dirichlet with weight functions)

For every $\Theta \in M_{m,n}$ and every $T \geq 1$, there is $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\begin{cases} \|\Theta_i \mathbf{q}\| \leq \alpha_i(1/T), & i = 1, \dots, m, \\ |q_j| \leq \beta_j(T), & j = 1, \dots, n. \end{cases}$$

Here Θ_i is the i th row of Θ .

Bad approximation with weight functions

Definition

We write $\Theta \in \mathbf{BA}_{\vec{\alpha}, \vec{\beta}}$ if there is $c > 0$ such that

$$\begin{cases} \|\Theta_i \mathbf{q}\| \leq \alpha_i(c/T), & i = 1, \dots, m, \\ |q_j| \leq \beta_j(T), & j = 1, \dots, n \end{cases}$$

has no solution $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ for all sufficiently large T .

The improvement parameter sits **inside** α_i .

For power weights, this recovers the usual weighted bad approximability.

Quasimultiplicativity

An increasing $h : I \rightarrow \mathbb{R}_{>0}$ is **quasimultiplicative (QM)** if there exist $R_0 > 1$ and $k_2 \geq k_1 > 0$ such that

$$R^{k_1} h(T) \leq h(RT) \leq R^{k_2} h(T)$$

whenever $R \geq R_0$ and $T, RT \in I$.

On each long enough multiplicative interval $[T, RT]$, the normalized growth $h(RT)/h(T)$ is bounded between two fixed powers of R .

We assume each α_i and each β_j is QM.

Dani correspondence along a curve

Define $F^{\vec{\alpha}, \vec{\beta}} = \{g(t) : t \geq 0\}$, where

$$g(t) = \text{diag}\left(\frac{1}{\alpha_1(e^{-t})}, \dots, \frac{1}{\alpha_m(e^{-t})}, \frac{1}{\beta_1(e^t)}, \dots, \frac{1}{\beta_n(e^t)}\right).$$

The product normalization gives $\det g(t) = 1$.

Theorem (Dani correspondence under QM)

$$\Theta \in \mathbf{BA}_{\vec{\alpha}, \vec{\beta}} \iff F^{\vec{\alpha}, \vec{\beta}} u_{\Theta} \mathbb{Z}^d \text{ is bounded in } X_d.$$

In general, $g(t+s) \neq g(t)g(s)$: the trajectory follows a curve, not a one-parameter subgroup.

Kleinbock–Neckrasov, [arXiv:2607.17021](https://arxiv.org/abs/2607.17021), Proposition 4.4. Neckrasov (2026), Proposition 3.4.

The logarithmic curve in $\mathfrak{a}$

Let A be the positive diagonal subgroup of $\mathrm{SL}_d(\mathbb{R})$, and $\mathfrak{a} = \mathrm{Lie}(A)$.

$$a_i(t) = -\log \alpha_i(e^{-t}), \quad b_j(t) = \log \beta_j(e^t).$$

$$\mathbf{v}(t) := \log g(t) = \mathrm{diag}(a_1(t), \dots, a_m(t), -b_1(t), \dots, -b_n(t)),$$

$$\sum_i a_i(t) = \sum_j b_j(t) = t.$$

For power weights, $\mathbf{v}(t)$ is a ray. General weight functions allow other curves.

The super-expanding cone for the block subgroup

The subgroup used in Diophantine approximation is

$$U = \{u_{\Theta} : \Theta \in M_{m,n}\} = \left\{ \begin{pmatrix} 1_m & \Theta \\ 0 & 1_n \end{pmatrix} : \Theta \in M_{m,n} \right\}.$$

Let us define its **super-expanding cone** by

$$\mathfrak{s}_u^+ = \{\text{diag}(a_1, \dots, a_m, -b_1, \dots, -b_n) \in \mathfrak{a} : a_i, b_j > 0\}.$$

For $\mathbf{v} = \text{diag}(a_1, \dots, a_m, -b_1, \dots, -b_n)$,

$$\exp(\mathbf{v})u_{\Theta}\exp(-\mathbf{v}) = u_{\Theta'}, \quad \Theta'_{ij} = e^{a_i+b_j}\Theta_{ij}.$$

Every matrix entry expands when $\mathbf{v} \in \mathfrak{s}_u^+$.

QM means uniform progress away from the walls

QM is equivalent to the existence of $\sigma, t_0 > 0$ such that

$$\begin{aligned} a_i(t + \tau) - a_i(t) &\geq \sigma\tau, \\ b_j(t + \tau) - b_j(t) &\geq \sigma\tau \end{aligned} \quad \text{for } t \geq 0, \tau \geq t_0.$$

Thus all sufficiently long increments lie in one smaller convex cone:

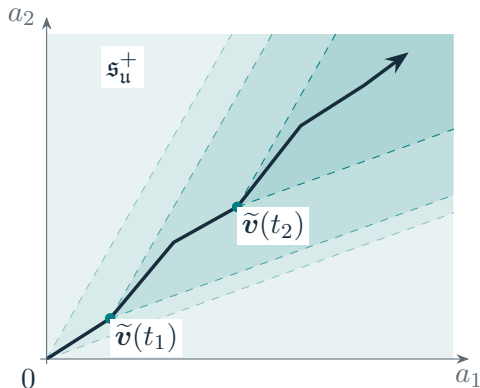
$$\boldsymbol{v}(t + \tau) - \boldsymbol{v}(t) \in \mathfrak{s}, \quad \bar{\mathfrak{s}} \subset \mathfrak{s}_u^+ \cup \{0\}.$$

Sample at kt_0 and interpolate. The resulting curve $\tilde{\boldsymbol{v}}$ stays a bounded distance from $\boldsymbol{v}$ and has the cone property **for every forward increment**.

Every tail stays in a translated smaller cone

The case $m = 2, n = 1$:

$$\mathbf{v}(t) = \text{diag}\left(a_1(t), a_2(t), -a_1(t) - a_2(t)\right).$$



One fixed cone $\mathfrak{s}$ works from every point:

$$\tilde{\mathbf{v}}([t, \infty)) \subset \tilde{\mathbf{v}}(t) + \bar{\mathfrak{s}}.$$

Here $\bar{\mathfrak{s}} \subset \mathfrak{s}_u^+ \cup \{0\}$.

The original QM curve stays a bounded distance from this interpolation.

Root directions associated with U

Let H be the product of the simple factors of G on which U projects nontrivially. Write $U = U_+(b)$, and let $P \subset H$ be a parabolic subgroup with unipotent radical U and $b \in P$. Choose a maximal connected Ad-diagonalizable $A \subset P$ containing b . Put

$$\mathfrak{h} = \text{Lie}(H), \quad \mathfrak{a} = \text{Lie}(A), \quad \mathfrak{u} = \text{Lie}(U).$$

For a choice of positive roots, let $\Phi_{\mathfrak{u}}$ consist of those whose root spaces occur in $\mathfrak{u}$. Then

$$\mathfrak{u} = \bigoplus_{\lambda \in \Phi_{\mathfrak{u}}} \mathfrak{h}_{\lambda}, \quad [\mathbf{v}, Y] = \lambda(\mathbf{v})Y \quad \text{for } Y \in \mathfrak{h}_{\lambda}.$$

Use the Killing inner product to identify roots (and, more generally, all linear functionals) with vectors:

$$\langle \mathbf{v}_{\lambda}, \mathbf{v} \rangle = \lambda(\mathbf{v}).$$

The algebraically expanding cone

$$\mathfrak{a}_u^+ := \left\{ \sum_{\lambda \in \Phi_u} t_\lambda \mathbf{v}_\lambda : t_\lambda > 0 \right\}, \quad A_U^+ = \exp(\mathfrak{a}_u^+).$$

Shi's representation-theoretic characterization

An element $a \in A$ belongs to A_U^+ precisely when

$$V^{\rho(U)} \subset V^+(\rho(a))$$

for every nontrivial finite-dimensional irreducible representation (ρ, V) of H .

Here $V^{\rho(U)}$ consists of U -fixed vectors: $\rho(u)w = w$ for all $u \in U$. And $V^+(\rho(a))$ is the sum of eigenspaces with eigenvalues of modulus > 1 .

The geometrically expanding cone

$$\mathfrak{e}_u^+ := \{\mathbf{v} \in \mathfrak{a} : \lambda(\mathbf{v}) > 0 \text{ for all } \lambda \in \Phi_u\}.$$

For $Y \in \mathfrak{h}_\lambda$,

$$\mathrm{Ad}(\exp \mathbf{v})Y = e^{\lambda(\mathbf{v})}Y.$$

Thus $\Omega_{\mathbf{v}}(u) := \exp(\mathbf{v})u \exp(-\mathbf{v})$ expands every root direction in U .

Equivalently, $E_U^+ := \exp(\mathfrak{e}_u^+) = \{a \in A : U \subset U_+(a)\}$.

Role in our proof: inverse conjugation shrinks tiles, allowing us to retain whole returning tiles in a Cantor construction.

Super-expansion combines both properties

$$\mathfrak{s}_u^+ := \mathfrak{a}_u^+ \cap \mathfrak{e}_u^+, \quad S_U^+ = A_U^+ \cap E_U^+.$$

Algebraic expansion

Equidistribution supplies enough returning mass.

For the Diophantine subgroup

$$U = \{u_\Theta\} = \left\{ \begin{pmatrix} I_m & \Theta \\ 0 & I_n \end{pmatrix} : \Theta \in M_{m,n}(\mathbb{R}) \right\},$$

write $\mathbf{v} = \text{diag}(a_1, \dots, a_m, -b_1, \dots, -b_n)$, with $\sum_i a_i = \sum_j b_j$. Then

$$\mathfrak{a}_u^+ = \{a_i, b_j > 0\}, \quad \mathfrak{e}_u^+ = \{a_i + b_j > 0 \ \forall i, j\}.$$

Here $\mathfrak{s}_u^+ = \mathfrak{a}_u^+ \subset \mathfrak{e}_u^+$. In general, neither cone need contain the other.

Geometric expansion

Conjugation turns that mass into whole returning tiles.

Super-expansion and effective equidistribution

For $v \in \mathfrak{s}_u^+$, write

$$[v]_{\mathfrak{s}_u^+} := \text{dist}(v, \partial \mathfrak{s}_u^+).$$

Theorem (R. Shi)

There exists $\zeta = \zeta(H, X, U, A) > 0$ such that, for every compact $Q \subset X$, $\varphi \in C_c^\infty(U)$ and $\psi \in C_c^\infty(X)$, there is $M = M(Q, \varphi, \psi) > 0$ with

$$\left| \int_U \varphi(u) \psi(\exp(v)ux) d\mu_U - \int_U \varphi d\mu_U \int_X \psi d\mu_X \right| \leq M e^{-\zeta [v]_{\mathfrak{s}_u^+}}$$

for all $x \in Q$ and $v \in \mathfrak{s}_u^+$.

Role in our proof: a large compact set receives a controlled proportion of an expanding piece of U .

Shi (2021), Theorem 1.5; Kleinbock–Neckrasov, [arXiv:2607.17021](https://arxiv.org/abs/2607.17021), Theorem 2.5 and Corollary 2.7.

Quasi-rays

Let $\mathfrak{s} \subset \mathfrak{a}$ be a convex cone with $\bar{\mathfrak{s}} \subset \mathfrak{s}_u^+ \cup \{0\}$.

Definition

A sequence $\mathcal{V} = (v_i)_{i \geq 1}$ in $\mathfrak{s}$ is a **quasi-ray** if, with $v_0 = 0$,

$$v_i - v_{i-1} \in \bar{\mathfrak{s}}, \quad 0 < c_0 \leq |v_i - v_{i-1}| \leq L < \infty.$$

A set $F \subset G$ is a **quasi-ray relative to $\mathfrak{s}$** if it lies in a bounded thickening of $\exp(\mathcal{V})$ for quasi-ray $\mathcal{V}$:

$$F \subset C \exp(\mathcal{V}) \quad \text{for some compact } C \subset G.$$

The main theorem for horospherical subgroups

Theorem (Kleinbock–Neckrasov)

Let G, Γ, U, A be as above. Suppose

$$\bar{\mathfrak{s}} \subset \mathfrak{s}_u^+ \cup \{0\}, \quad F \text{ is a quasi-ray relative to } \mathfrak{s}.$$

Then for every $x \in X = G/\Gamma$,

$$\{u \in U : ux \in \mathcal{B}(F)\} \text{ is thick in } U.$$

Equivalently, for every nonempty open $W \subset U$,

$$\dim_H \{u \in W : Fux \text{ is bounded}\} = \dim U.$$

The main theorem on the whole space

Under the same assumptions,

$$\mathcal{B}(F) = \{x \in X : Fx \text{ is bounded}\} \text{ is thick in } X.$$

A brief slicing argument

Choose a submanifold $V \subset G$ through the identity with $T_e G = T_e U \oplus T_e V$. For small neighborhoods $O_U \subset U$ and $O_V \subset V$ of the identity, the map

$$O_U \times O_V \longrightarrow X, \quad (u, g) \longmapsto ugx$$

is a local product chart. Every slice over $g \in O_V$ has dimension $\dim U$ by the theorem for U , applied to gx .

$$\dim_H \mathcal{B}(F) \text{ locally} \geq \dim O_V + \dim U = \dim X.$$

The proof target: Theorem 3.1

For a quasi-ray $\mathcal{V} = (v_i)$ in $\mathfrak{s}_u^+$, write

$$\rho_{\mathfrak{s}_u^+}(\mathcal{V}) := \inf_{i \geq 1} \lfloor v_i - v_{i-1} \rfloor_{\mathfrak{s}_u^+}.$$

Given $\varepsilon > 0$, $x \in X$ and nonempty open $W \subset U$, find a compact K and $t > 0$ such that, if $\rho_{\mathfrak{s}_u^+}(\mathcal{V}) \geq t$,

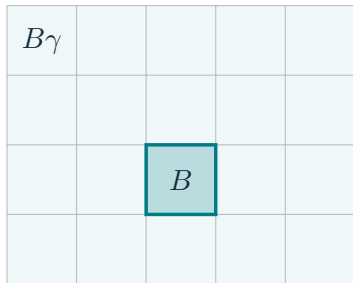
$$\dim_H \{u \in W : \exp(v_i)ux \in K \text{ for every } i \geq 1\} \geq \dim U - \varepsilon.$$

For our strict-subcone quasi-ray, choosing a subsequence makes t arbitrarily large.

The omitted points stay within bounded distance, so compactness of trajectories is preserved after enlarging K .

Ingredient 1: geometric expansion

A tessellation of U



A square model for (B, Λ)

Choose a small tessellation (B, Λ) of U , with $e \in B$ and $\text{diam } B \leq r$.

$$U = \bigcup_{\gamma \in \Lambda} \overline{B\gamma}.$$

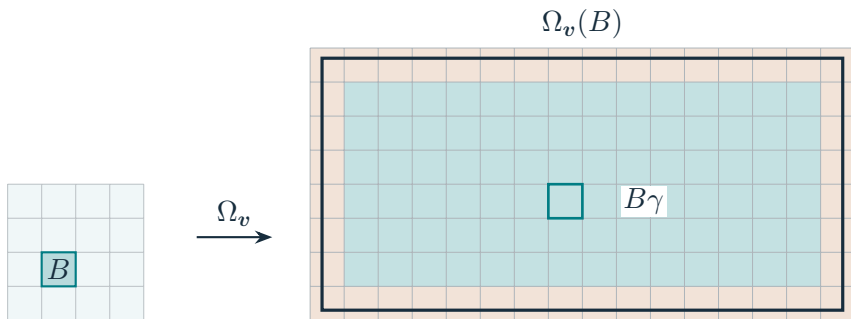
The open tiles $B\gamma$ are disjoint and

$$\mu_U(\partial B) = 0.$$

We start with $\mathcal{E}_0 = \{\overline{B}\}$ and subdivide it into smaller blocks.

Ingredient 1: geometric expansion

Expand, then count whole tiles



$$\Omega_v(u) = \exp(\mathbf{v})u \exp(-\mathbf{v}), \quad J_v = \frac{\mu_U(\Omega_v(B))}{\mu_U(B)}.$$

Teal: whole tiles. Orange: possible boundary loss.

Ingredient 1: geometric expansion

Tiles contract exponentially

If $\bar{s}$ is a closed cone with

$$\bar{s} \setminus \{0\} \subset \mathfrak{e}_u^+, \quad v \in \bar{s},$$

then $\lfloor v \rfloor_{\mathfrak{e}_u^+} \geq c_e |v|$, and

$$\ell_v := \text{Lip}(\Omega_v^{-1}|_U) \leq \kappa e^{-c \lfloor v \rfloor_{\mathfrak{e}_u^+}} \longrightarrow 0 \quad (|v| \rightarrow \infty).$$

Corollary

For large enough $\lfloor v \rfloor_{\mathfrak{e}_u^+}$,

$$\#\{\gamma \in \Lambda : B\gamma \subset \Omega_v(B)\} \geq J_v(1 - \delta).$$

After pullback, these tiles cover at least $1 - \delta$ of the volume of B .

Here we only use geometric expansion: the property of $\mathfrak{e}_u^+$.

Ingredient 2: return to a compact set

Compact return from equidistribution

Choose compact K with $0 < \mu_X(K) < 1$ and $\mu_X(\partial K) = 0$, and any compact $Q \subset X$. For a fixed closed cone $\bar{\mathfrak{s}}$ with

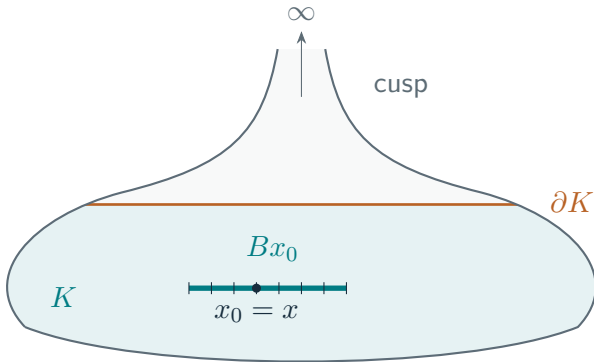
$$\bar{\mathfrak{s}} \setminus \{0\} \subset \mathfrak{s}_u^+, \quad \mathbf{v} \in \bar{\mathfrak{s}}, \quad |\mathbf{v}| \longrightarrow \infty,$$

the super-expanding wall-distance tends to infinity. For the tile B ,

$$\frac{\mu_U\{u \in B : \exp(\mathbf{v})ux \in K\}}{\mu_U(B)} \longrightarrow \mu_X(K), \quad \text{uniformly for } x \in Q.$$

Ingredient 2: return to a compact set

A small horospherical piece inside K



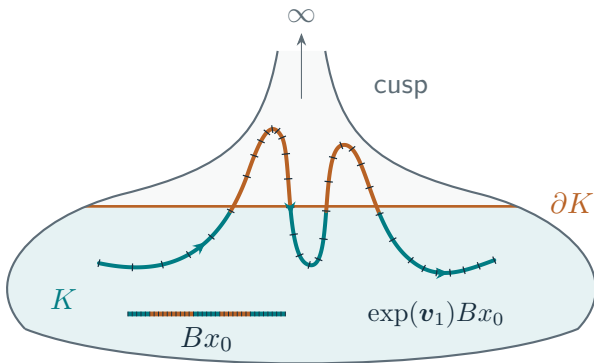
Choose $x_0 \in K$ and a small tile B .

The short arc represents Bx_0 .

Its subdivisions are inverse images of the tiles used after expansion.

Ingredient 2: return to a compact set

The expanded piece makes excursions and returns



$$\begin{aligned} & \exp(v_1)Bx_0 \\ &= \Omega_{v_1}(B) \exp(v_1)x_0. \end{aligned}$$

Teal pieces lie wholly in K .

Orange pieces leave K or cross its boundary.

Returning *mass* has proportion close to $\mu_X(K)$.

Ingredient 2: return to a compact set

From returning mass to whole returning tiles

Define

$$\Lambda_v(x, K) = \{\gamma \in \Lambda : B\gamma \subset \Omega_v(B), \ B\gamma \exp(\mathbf{v})x \subset K\}.$$

Proposition

For $0 < \eta < \mu_X(K)$, choose B sufficiently small. Then for all $x \in K$ and sufficiently large $\lfloor \mathbf{v} \rfloor_{\mathfrak{s}_u^+}$,

$$\#\Lambda_v(x, K) \geq J_v(\mu_X(K) - \eta) = \frac{\mu_U(\Omega_v(B))}{\mu_U(B)} (\mu_X(K) - \eta).$$

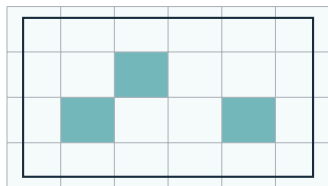
Taking $\eta = \mu_X(K) - \mu_X(K)^2$ gives the lower bound $J_v \mu_X(K)^2$.

The first generation

Keep only the good tiles and pull them back:

$$\mathcal{E}_1 = \{\Omega_{v_1}^{-1}(\overline{B}\gamma) : \gamma \in \Lambda_{v_1}(x, K)\}.$$

Good tiles in $\Omega_{v_1}(B)$



$$\xrightarrow{\Omega_{v_1}^{-1}}$$

Children in $\overline{B}$



Every child E satisfies

$$E \subset \overline{B}, \quad \exp(v_1)Ex \subset K.$$

Number of children: $\#\mathcal{E}_1 \geq J_{v_1}(\mu_X(K) - \eta).$

The same construction inside every parent

For a parent $E \in \mathcal{E}_i$, write

$$E = \Omega_{\mathbf{v}_i}^{-1}(\overline{B})u, \quad y_i = \exp(\mathbf{v}_i)ux \in K.$$

At the next step use the **increment** $\mathbf{v}_{i+1} - \mathbf{v}_i$:

$$\gamma \in \Lambda_{\mathbf{v}_{i+1}-\mathbf{v}_i}(y_i, K).$$

The corresponding child is

$$E_\gamma = \Omega_{\mathbf{v}_{i+1}}^{-1}(\overline{B}\gamma)u \subset E.$$

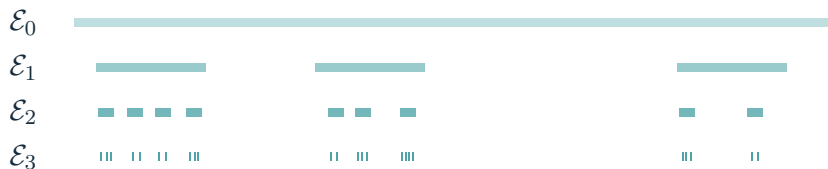
It satisfies

$$\exp(\mathbf{v}_{i+1})E_\gamma x = \overline{B}\gamma \exp(\mathbf{v}_{i+1} - \mathbf{v}_i)y_i \subset K.$$

Number of children:

$$\#\mathcal{E}_{i+1}(E) \geq J_{\mathbf{v}_{i+1}-\mathbf{v}_i}\mu_X(K)^2.$$

A Cantor-like set of bounded trajectories



The construction is **strongly tree-like**: nested compact tiles, disjoint up to null boundaries, nonempty descendants, and diameters tending to zero.

$$E_\infty = \bigcap_{i \geq 1} \bigcup_{E \in \mathcal{E}_i} E.$$

For $u \in E_\infty$, every generation enforces another return:

$$\exp(\mathbf{v}_i)ux \in K \quad \text{for every } i \geq 1.$$

Ingredient 3: lower bound for dimension

The density retained inside a parent

For $E \in \mathcal{E}_i$, define

$$\text{density}(\mathcal{E}_{i+1}, E) = \frac{\mu_U\left(\bigcup \mathcal{E}_{i+1}(E)\right)}{\mu_U(E)}, \quad \Delta_i = \inf_{E \in \mathcal{E}_i} \text{density}(\mathcal{E}_{i+1}, E).$$

Every child has $1/J_{v_{i+1}-v_i}$ of its parent's volume. Hence

$$\text{density}(\mathcal{E}_{i+1}, E) = \frac{\#\Lambda_{v_{i+1}-v_i}(y_i, K)}{J_{v_{i+1}-v_i}} \geq \mu_X(K)^2.$$

Ingredient 2 gives exactly the density bound: $\Delta_i \geq \mu_X(K)^2$ for every i .

Ingredient 3: lower bound for dimension

Let $d_k = \max\{\text{diam } E : E \in \mathcal{E}_k\}$ be the largest tile diameter at level k .

Lemma (Urbański)

For a strongly tree-like construction with $d_k \rightarrow 0$,

$$\dim_H E_\infty \geq \dim U - \limsup_{k \rightarrow \infty} \frac{\sum_{i=0}^k \log \Delta_i}{\log d_k}.$$

Here $\Delta_i \geq \mu_X(K)^2$ and $d_i \leq r\kappa^i e^{-ict}$. Thus, for $ct > \log \kappa$,

$$\dim_H E_\infty \geq \dim U - \frac{2 \log(1/\mu_X(K))}{ct - \log \kappa}.$$

Weights $(1, 0)$: a direction on a wall

For simultaneous approximation of (x, y) , a zero weight leaves y unrestricted:

$$\mathbf{BA}(1, 0) = \left\{ (x, y) : \inf_{q \in \mathbb{N}} q \|qx\| > 0 \right\} = \mathbf{BA}_{1,1} \times \mathbb{R}.$$

The corresponding diagonal flow and unipotent matrices are

$$g_t = \text{diag}(e^t, 1, e^{-t}), \quad u_{x,y} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $\text{diag}(1, 0, -1) \in \partial \mathfrak{s}_u^+$, and

$$\left\{ (x, y) : \{g_t u_{x,y} \mathbb{Z}^3 : t \geq 0\} \text{ is bounded} \right\} = \mathbf{BA}_{1,1} \times \mathbb{R}.$$

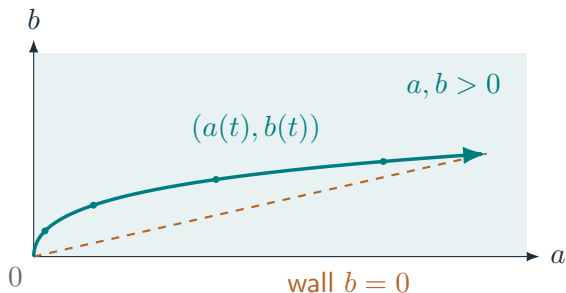
A theorem allowing wall directions would recover thickness in $\mathbb{R}^2$.

Example 4.5: a logarithmic curve near a wall

In $\mathrm{SL}_4(\mathbb{R})$, consider

$$g(t) = \mathrm{diag}\left(\frac{e^t}{1+t}, 1+t, \frac{1+t}{e^t}, \frac{1}{1+t}\right), \quad t \geq 0.$$

Write $\log g(t) = \mathrm{diag}(a(t), b(t), -a(t), -b(t))$.



$$a(t) = t - \log(1+t),$$

$$b(t) = \log(1+t).$$

$$\frac{b(t)}{a(t) + b(t)} \longrightarrow 0,$$

$$b(t+1) - b(t) \longrightarrow 0.$$

No fixed strict subcone contains all forward increments.

Example 4.5: bad approximation without QM

Put $\ell(T) = 1 + \log T$. The weight functions satisfy

$$\alpha_1(1/T) = \frac{\ell(T)}{T}, \quad \alpha_2(1/T) = \frac{1}{\ell(T)}, \quad \beta_1(T) = \frac{T}{\ell(T)}, \quad \beta_2(T) = \ell(T).$$

Define $\mathbf{BA}_{\vec{\alpha}, \vec{\beta}}^{\text{dyn}}$ by requiring, for some $\delta > 0$, that

$$\begin{cases} \|\Theta_i \mathbf{q}\| \leq \delta \alpha_i(1/T), & i = 1, 2, \\ |q_j| \leq \delta \beta_j(T), & j = 1, 2 \end{cases}$$

have no $\mathbf{q} \in \mathbb{Z}^2 \setminus \{0\}$ solution for all large T .

With constants **outside**, Mahler gives the bounded-trajectory condition.

A theorem for the full closed cone would give

$$\dim_H \left(\mathbf{BA}_{\vec{\alpha}, \vec{\beta}}^{\text{dyn}} \cap V \right) = 4 \quad \text{for every nonempty open } V \subset M_{2,2}.$$

Allowing directions on the walls

Ongoing project

Allow increments in $\overline{s_u^+}$, including its faces. The goal is again thickness of bounded trajectories.

What can fail?

- Ingredient 1: expansion in every direction.
- Ingredient 2: Haar equidistribution.
- Ingredient 3: rapid decay of diameters.

Ingredient 1: tiles adapted to neutral directions

The issue at a wall

Some directions no longer expand. The earlier boundary estimate fails, and poorly adapted tiles may cover a vanishing proportion of the expanded block.

Proposed replacement: Malcev boxes

Use ordered exponential coordinates adapted to the Lie brackets. Align the tiles in the neutral directions.

Count whole tiles in the expanding fibers, then lift them to the group while controlling the commutator terms.

The target is a uniform positive fraction of whole tiles.

Ingredient 2: return without Haar equidistribution

An algebraic wall can preserve an obstruction

In X_3 , take

$$U = \left\{ \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} : s, r \in \mathbb{R} \right\}, \quad a_Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^Q & 0 \\ 0 & 0 & e^{-Q} \end{pmatrix}.$$

Every lattice $a_Q u \mathbb{Z}^3$ contains e_1 .

Choose compact sets adapted to cusp escape

For lattices: put compatible lower bounds on covolumes of primitive sublattices of *every rank*.

For general G/Γ : use linear cusp tests in representations. The thresholds are chosen so that the bounds already secured remain protected at the next step.

Ingredient 2: whole-box return

For the adapted exhaustion $K_\delta \nearrow X$ as $\delta \downarrow 0$, put

$$K_\delta^B = \{x : Bx \subset K_\delta\}.$$

Choose an adapted Malcev tessellation $\{B\gamma : \gamma \in \Lambda\}$. Write $v = Q\lambda$, where $\lambda \in \overline{\mathfrak{s}_u^+}$ is a unit vector and every defining cone functional is zero or at least $c > 0$ on λ .

Proposed whole-box return (work in progress)

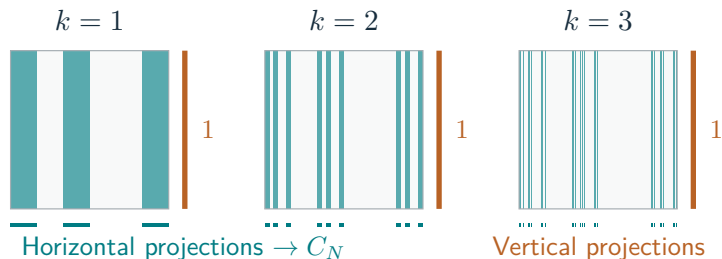
There is $\alpha > 0$ such that, for sufficiently small δ and sufficiently large Q , uniformly over these λ and $x \in K_\delta^B$,

$$\# \left\{ \gamma \in \Lambda : \begin{array}{l} B\gamma \subset \exp(v)B \exp(-v), \\ B\gamma \exp(v)x \subset K_\delta \end{array} \right\} \geq \alpha \frac{\mu_U(\exp(v)B \exp(-v))}{\mu_U(B)}.$$

Each retained tile supplies a new center $\gamma \exp(v)x \in K_\delta^B$.

Ingredient 3: a neutral direction keeps its dimension

Fix $0 < \alpha < 1$ and large N . Keep $M = \lceil \alpha N \rceil$ of N equal horizontal subintervals, with the full vertical interval at every step.

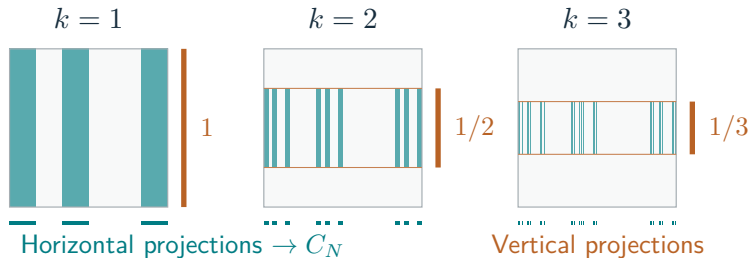


$$E_\infty = C_N \times [0, 1], \quad \dim_H E_\infty = \frac{\log M}{\log N} + 1 \xrightarrow{N \rightarrow \infty} 2.$$

$d_k \geq 1$: Urbański's shrinking-diameter hypothesis fails.

A slow contraction can remove an entire dimension

Use the **same horizontal selection**, but give every level- k rectangle the common centered vertical interval I_k of height $1/k$.



$$E_\infty = C_N \times \left\{ \frac{1}{2} \right\}, \quad \dim_H E_\infty = \frac{\log M}{\log N} + 0 \xrightarrow{N \rightarrow \infty} 1.$$

$d_k \asymp 1/k$: Urbański's lemma gives only $\dim_H E_\infty \geq 0$.

Ingredient 3: dimension at separate directional scales

Proposed lemma (work in progress)

Use compatible nested Malcev boxes in U , with neutral projections aligned, and retain at least a fixed fraction α of eligible children.

Fix large Q and $0 < c < 1$. Each side length must satisfy

$$\frac{d_j^{(k+1)}}{d_j^{(k)}} \in \underbrace{\{1\}}_{\text{neutral}} \cup \underbrace{[e^{-Q}, e^{-cQ}]}_{\text{definite contraction}}.$$

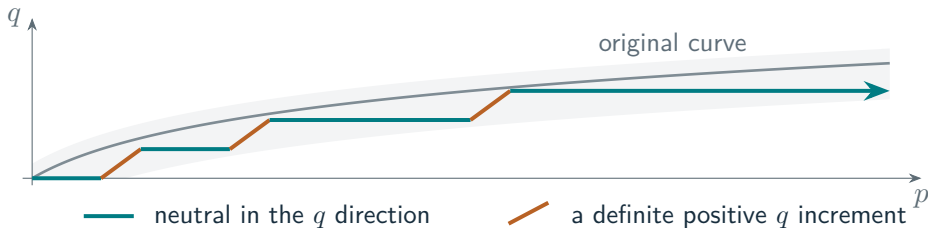
At least one direction contracts at each step.

$$\dim_H E_\infty \geq \dim U \left(1 - \frac{C}{Q}\right), \quad C = C(\alpha, c, U).$$

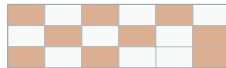
Replacing tiny steps by neutral stretches and bursts

Approximate the logarithmic curve within bounded displacement by a polygonal path whose unit directions satisfy

$$\ell_j(\lambda) = 0 \quad \text{or} \quad \ell_j(\lambda) \geq c \quad \text{for every defining cone functional } \ell_j.$$



Neutral step: keep the vertical fiber.



Active step: branch in both directions.

The boundary problem

The three proposed replacements fit the same induction

1. Malcev boxes retain whole tiles despite neutral directions.
2. Adapted compact sets give repeated return of a positive proportion of those tiles.
3. Directional mass estimates preserve dimension.

Goal: thickness for quasi-rays in the full closed super-expanding cone.

References

- D. Kleinbock and V. Neckrasov, *Bounded trajectories of quasi-rays on homogeneous spaces and Diophantine approximation with weight functions*. [arXiv:2607.17021](#) (2026).
- D. Kleinbock and G. A. Margulis, *Bounded orbits of nonquasiunipotent flows on homogeneous spaces*. Amer. Math. Soc. Transl. (2) **171** (1996), 141–172.
- S. G. Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*. J. Reine Angew. Math. **359** (1985), 55–89.
- W. M. Schmidt, *Badly approximable systems of linear forms*. J. Number Theory **1** (1969), 139–154.
- R. Shi, *Expanding cone and applications to homogeneous dynamics*. Int. Math. Res. Not. IMRN **2021**, 7060–7095.
- V. Neckrasov, *Khintchine-type theorems for weighted uniform inhomogeneous approximations via transference principle*. Mathematika **72** (2026), e70089.
- M. Urbański, *The Hausdorff dimension of the set of points with nondense orbit under a hyperbolic dynamical system*. Nonlinearity **4** (1991), 385–397, Lemma 2.1.
- Classical metric results: A. Khintchine, Math. Ann. **92** (1924), 115–125; Math. Z. **24** (1926), 706–714. A. Groshev, Dokl. Akad. Nauk SSSR **19** (1938), 151–152.